# The Root Distribution of Power Series with Prime–Composite Indicators and a New Irrational Constant: "Descartes's Prime Constant" (D)

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**Abstract**

This study defines an infinite power series $f(x) = \Sigma\, \chi(n)x^n$, whose coefficients take the value +1 or −1 according to whether successive natural numbers are prime or composite. By examining the boundary behavior of this function on the interval (0,1), the existence of at least one root in this interval is proven via the Intermediate Value Theorem, and the smallest such root is defined as "Descartes's Prime Constant" ($D$). The behavior as $x \to 1^-$ is rigorously established using the Prime Number Theorem and Karamata's Tauberian Theorem; the numerical value of the constant has been redetermined via high-precision computation as $D \approx 0.8712951466$. Given a logical gap in the original proof approach, the irrationality of $D$ is presented as an open conjecture supported by computational evidence.

**Keywords**

Descartes' Rule of Signs; Prime Number Theorem; Power Series; Analytic Roots; Irrationality; Tauberian Theorems.

**Öz**

Bu çalışmada, katsayıları ardışık doğal sayıların asal veya bileşik olma durumuna göre +1 veya −1 değerlerini alan sonsuz bir kuvvet serisi $f(x) = \Sigma\, \chi(n)x^n$ tanımlanmıştır. Fonksiyonun (0,1) aralığındaki uç davranışları incelenerek, bu aralıkta en az bir kökün varlığı ispatlanmış ve bu köklerin en küçüğü "Descartes'ın Asal Sabiti" ($D$) olarak tanımlanmıştır. Sabitin sayısal değeri $D \approx 0.8712951466$ olarak yeniden belirlenmiş; irrasyonelliği ise açık bir sanı olarak sunulmuştur.

**Anahtar Kelimeler**

Descartes İşaret Kuralı; Asal Sayı Teoremi; Kuvvet Serileri; Analitik Kökler; İrrasyonalite; Tauberian Teoremler.

## 1. Introduction

The bounding of polynomial roots, one of the fundamental problems of algebra, was given a deterministic structure by the "Rule of Signs" set forth by René Descartes in his 1637 work *La Géométrie* (Descartes, 1637). Descartes' Rule of Signs states that the number of positive roots of a real-coefficient polynomial $P(x)$ cannot exceed the number of sign changes in its sequence of coefficients (Aslan, 2001).

When this rule is applied to infinite-degree power series with coefficients drawn from an aperiodic distribution over ±1 (integers), the behavior of the function on the interval (0,1) presents a considerably complex topological structure. The distribution of primes lies at the heart of the Prime Number Theorem, whose foundations trace back to Hadamard's (1896) work on the zeros of the $\zeta(s)$ function, and constitutes one of the most aperiodic sequences in number theory. In this study, the set of Prime Numbers (**P**) is used as the coefficient generator, and the convergence of these chaotic sign changes to an analytic constant is examined. To our knowledge, no study in the literature has directly examined the root structure of such "prime-indicator" power series; this constitutes the original contribution of the present work.

The remainder of the paper is organized as follows. Section 2 formally defines the indicator function and the power series, and determines the radius of convergence. Section 3 examines the boundary behaviors and proves the existence of the constant $D$. Section 4 addresses the question of irrationality as a conjecture. Section 5 discusses open problems.

## 2. Definition of the Indicator Function and the Power Series

*Definition 2.1 (Prime–Composite Indicator).* For every integer $n \geq 2$, the coefficient function $\chi(n)$ is defined as follows:

$$\chi(n) = +1, \quad \text{if } n \in \mathbf{P} \text{ (prime)}$$

$$\chi(n) = -1, \quad \text{if } n \in \mathbf{C} \text{ (composite)}$$

*Remark.* Since 1 is neither classified as prime nor composite, the series begins at $n = 2$.

*Definition 2.2 (Prime Power Series).* The analytic function $f(x)$, using the coefficients $\chi(n)$, is defined as:

$$f(x) = \sum_{n=2}^{\infty} \chi(n)x^n = x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 - x^9 - \cdots$$

*Proposition 2.3 (Radius of Convergence).* Since $|\chi(n)| = 1$ for all values of $n$, $\limsup_{n\to\infty} |\chi(n)|^{1/n} = 1$. By the Cauchy–Hadamard theorem, the radius of convergence is $R = 1/1 = 1$. The function converges absolutely on the open interval $x \in (-1, 1)$.

## 3. Existence of the Constant and Asymptotic Behavior

To show that $f$ has a root on the interval (0, 1), the boundary behaviors of the function are examined.

*Lemma 3.1 (Behavior Near Zero).* As $x \to 0^+$, the asymptotic behavior of the function is $f(x) \approx x^2 > 0$, since the dominant term of the series is $\chi(2)x^2 = +x^2$ (as 2 is prime). Hence the function starts on the positive side: $f(0^+) > 0$.

*Lemma 3.2 (Behavior Near One).* As $x \to 1^-$, the function diverges to negative infinity: $\lim_{x\to1^-} f(x) = -\infty$.

**Proof.** The function $f(x)$ can be split into prime and composite sums:

$$f(x) = \Sigma_{p\in P}\, x^p - \Sigma_{c\in C}\, x^c$$

Using the relation $\Sigma_{n=2}^{\infty} x^n = \Sigma_p x^p + \Sigma_c x^c$ for the sum over all $n \geq 2$, we obtain $\Sigma_c x^c = \Sigma_{n=2}^{\infty} x^n - \Sigma_p x^p$. Substituting this into the definition of $f(x)$:

$$f(x) = 2\, \Sigma_p\, x^p - \Sigma_{n=2}^{\infty}\, x^n$$

The behavior of the first sum is straightforward: $\Sigma_{n=2}^{\infty} x^n = x^2/(1-x)$, so as $x \to 1^-$ this term diverges on the order of $\approx 1/(1-x)$.

For the second sum, by the Prime Number Theorem (Hadamard, 1896), the prime-counting function satisfies $\pi(N) \approx N/\ln N$. Karamata's Tauberian Theorem (Karamata, 1930; see also Korevaar, 2004) relates the $x \to 1^-$ behavior of the power series of a non-negative sequence with partial sums $A(N) \approx N^\rho L(N)$ (where $L$ is slowly varying) to this asymptotic. Taking $\rho = 1$ and $L(N) = 1/\ln N$ for $\pi(N) \approx N/\ln N$, we obtain:

$$\Sigma_p\, x^p \approx 1 / [(1-x) \ln(1/(1-x))] \quad (x \to 1^-)$$

This expression also diverges to infinity, but grows *more slowly* than $1/(1-x)$ by a logarithmic factor (owing to the thinning effect on primes examined by Hardy & Littlewood, 1914). Combining these to obtain $f(x)$:

$$f(x) = 2\, \Sigma_p\, x^p - \Sigma_{n=2}^{\infty}\, x^n \approx [1/(1-x)] \cdot [2/\ln(1/(1-x)) - 1]$$

As $x \to 1^-$, $\ln(1/(1-x)) \to +\infty$, so the term $2/\ln(1/(1-x)) \to 0$, and the bracketed expression converges to $-1$. Therefore:

$$f(x) \approx -1/(1-x) \to -\infty \quad (x \to 1^-)$$

This result shows that the (negatively signed) contribution from the composite numbers dominates the contribution from the primes — because, by the Prime Number Theorem, the primes "thin out" by a logarithmic factor. ■

## 4. Definition of the Constant and the Irrationality Conjecture

*Theorem 4.1 (Existence of Descartes's Prime Constant).* By the Intermediate Value Theorem together with Lemmas 3.1 and 3.2, since $f(0^+) > 0$ and $\lim_{x\to 1^-} f(x) = -\infty$, and since $f$ is continuous on (0, 1), there exists $x \in (0, 1)$ such that $f(x) = 0$.

*Definition 4.2 (The Constant D).* The smallest positive real root satisfying $f(x) = 0$ is defined as Descartes's Prime Constant ($D$):

$$D := inf\{x \in (0, 1) \,/\, f(x) = 0\}$$

*Remark 4.3 (Corrected Numerical Value).* The function has been recomputed using exact-precision arithmetic (with primes generated via the Sieve of Eratosthenes, and stability tested between 4,000 and 16,000 terms). The scan confirms that $f$ changes sign exactly once on the interval (0, 1) (reaching a local maximum near $x \approx 0.74$ before decreasing and crossing zero); this unique sign change yields the following precise value:

$$D \approx 0.8712951466350017\ldots$$

It has been confirmed that $f$ remains positive on the sub-interval (0, 0.86) and crosses zero only near $x \approx 0.871$; this finding provides numerical support for the uniqueness question addressed in Open Problem 2 (Section 5).

*Conjecture 4.4 (The Irrationality Conjecture).* It is conjectured that Descartes's Prime Constant ($D$) is not a rational number ($D \notin \mathbf{Q}$).

*Heuristic Argument.* For a power series with integer (±1) coefficients to be expressed as a rational function $f(x) = P(x)/Q(x)$, its coefficient sequence must eventually be linear-recursive (Stanley, 1997); for bounded (±1-valued) sequences, this generally corresponds to the sequence eventually being periodic. However, the sequence $\chi(n)$ depends on the primes, and by Dirichlet's Theorem on Arithmetic Progressions together with known results on prime gaps, there exists no infinite arithmetic progression consisting solely of primes or solely of composites; hence $\chi(n)$ is aperiodic, and $f(x)$ cannot be a rational function.

*Important Caveat (Logical Limitation).* The fact that $f$ is not a rational function does **not directly prove** that its root $D$ is irrational; this is an inference tacitly assumed in the original study but not generally valid (a non-rational analytic function may still have rational roots; for instance, the transcendental function $g(x) = e^x - 2e^{1/2}$ vanishes at the rational point $x = 1/2$). For this reason, the irrationality of $D$ is presented in this study not as a proven theorem but as a strong conjecture. Circumstantial evidence supporting the conjecture includes: (i) no discernible periodic pattern is observed in the decimal expansion of $D$ (computed above to 10 digits); (ii) the value of $D$, while close to simple low-denominator fractions (1/2, 4/5, 6/7, 7/8, etc.) or known quadratic algebraic numbers ($\sqrt{3}/2 \approx 0.866$, etc.), is not equal to any of them. These findings support the conjecture but do not constitute a rigorous proof.

## 5. Discussion and Open Problems

The newly defined constant $D$ offers a strong foundation for investigating the asymptotic bridges between sign sequences and the roots of analytic functions in number theory.

**Open Problem 1 (Rigorous Proof of the Irrationality Conjecture).** Can a rigorous proof be given for the claim $D \notin \mathbf{Q}$ proposed in Conjecture 4.4? This would require an independent Diophantine approximation argument to close the logical gap between the non-rationality of $f$ and the irrationality of $D$.

**Open Problem 2 (Uniqueness of the Root).** Although Descartes' Rule of Signs shows that the coefficient sequence has infinitely many sign changes, this formal bound does not directly apply to the number of roots (the rule provides only an upper bound). Numerical scanning shows that $f$ remains positive on $(0, 0.86)$ and crosses zero exactly once at $D$; however, $f$ is not monotonic on this interval (it reaches a local maximum near $x \approx 0.74$). A rigorous proof that $D$ is the *unique* root on $(0, 1)$ remains open.

**Open Problem 3 (Transcendence).** Whether $D$ is a transcendental number that is not algebraic has not yet been investigated in the context of the Lindemann–Weierstrass or Baker theorems.

## 6. Conclusion

In this study, the spirit of Descartes' Rule of Signs has been combined with the aperiodic structure of the distribution of primes to define an analytic equilibrium point, Descartes's Prime Constant ($D$). The existence of the constant has been rigorously proven via the Intermediate Value Theorem; the boundary behavior as $x \to 1^-$ has been established through a complete asymptotic analysis using the Prime Number Theorem and Karamata's Tauberian Theorem. The numerical value has been redetermined via high-precision computation as $D \approx 0.8712951466$, which differs substantially from the original estimate, underscoring the importance of subjecting a study's computational foundations to independent verification. The irrationality of the constant is presented not as a claim without full logical support, but as an open conjecture backed by computational evidence.

## Ethics Statement and Additional Information

**Ethics Committee Approval:** This study is a theoretical/analytical mathematics investigation that does not involve any experimentation on human or animal subjects; therefore, ethics committee approval is not required.

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