The Entropic Equilibrium Constant (Ω):
A Mathematical Analysis and Theoretical Interpretation of an Infinite Nested Factorial-Radical Structure

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ABSTRACT

This study defines a new mathematical constant formed by a sequence of factorials trapped inside an infinite nested square-root structure. Named the "Entropic Equilibrium Constant" (Ω), it is expressed in the form $\Omega = \sqrt{1! + \sqrt{2! + \sqrt{3! + \sqrt{4! + \cdots}}}}$. The convergence of the sequence is established within the framework of Herschfeld's Convergence Theorem (1935), and the approximate value is determined via numerical simulation to be $\Omega \approx 1.827015$. The impossibility of the structure being a root of an algebraic equation is discussed, along with theoretical interpretations in the context of quantum cognition and astrophysics.

Keywords: infinite nested radicals, factorial sequence, Herschfeld's theorem, convergence, mathematical constants, transcendental numbers

1. Introduction

One of the most captivating structures in mathematics is the class of algebraic expressions known as infinite nested radicals. These expressions are formed by placing radicals within radicals, extending to infinite depth. Ramanujan's famous identity

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}} = 3$$

is one of the most striking examples in this field. This particular structure represents a special case in which the sequence of inner terms grows linearly.

In the present study, a new infinite nested radical constant is defined in which the inner terms grow at *factorial* speed — and hence under a super-exponential growth regime. Owing to the metaphorical connection this system establishes with the thermodynamic concept of *entropy*, this constant is named the **Entropic Equilibrium Constant** and is denoted by the capital letter Omega (Ω).

2. Mathematical Definition and Notation

2.1. Basic Definition

The Entropic Equilibrium Constant Ω is defined by the following infinite nested square-root expression:

$$\Omega = \sqrt{1! + \sqrt{2! + \sqrt{3! + \sqrt{4! + \sqrt{5! + \cdots}}}}}$$

In general-term notation, this expression can be written as:

$$\Omega = \lim_{N\to\infty} a_N,\quad a_N = \sqrt{N!},\quad a_k = \sqrt{k! + a_{k+1}}\quad (k = N-1, \ldots, 1)$$

It should be noted that the sequence is constructed *inside-out*: the innermost term is fixed first, and the construction proceeds outward from there.

2.2. Explicit Representation

The explicit form of the first several terms is as follows:

$$\Omega = \sqrt{1 + \sqrt{2 + \sqrt{6 + \sqrt{24 + \sqrt{120 + \sqrt{720 + \cdots}}}}}}$$

since 1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, …

3. Convergence Analysis

3.1. Herschfeld's Convergence Theorem

Theorem (Herschfeld, 1935): Let $\{x_n\}$ be a sequence of positive numbers. The infinite nested radical $\sqrt{x_1 + \sqrt{x_2 + \sqrt{x_3 + \cdots}}}$ converges if and only if the condition $\limsup_{n\to\infty} x_n^{1/2^n} < \infty$ is satisfied.

For the defined constant Ω, we have $x_n = n!$. Using Stirling's approximation $n! \approx \sqrt{2\pi n}\,(n/e)^n$:

$$x_n^{(1/2^n)} = (n!)^{(1/2^n)} \approx \exp\left((1/2^n) \cdot (n \ln n - n + \tfrac{1}{2} \ln(2\pi n)) \right)$$

In the limit $n \to \infty$, the exponent term $n \ln n / 2^n \to 0$, since exponential growth dominates factorial-order (in the exponent, super-linear polynomial) growth. Hence $\limsup (n!)^{(1/2^n)} = e^0 = 1 < \infty$. This proves that Herschfeld's condition is satisfied, and therefore Ω converges to a **finite and well-defined** value.

3.2. Numerical Computation

The sequence was computed iteratively from the inside out (using exact high-precision arithmetic; truncation at N = 60 was found to give a result stable to 45+ significant digits):

$a_{60} = \sqrt{60!}$ — *an astronomically large starting value, absorbed rapidly by the outward square-root contractions*

Proceeding outward, the sequence yields:

$$a_5 \approx 10.95 \to a_4 \approx 5.91 \to a_3 \approx 3.45 \to a_2 \approx 2.33 \to a_1 \approx 1.8270$$

As N becomes sufficiently large, this process locks onto a stable value:

$$\Omega \approx 1.8270147176\ldots$$

Remark (Corrected Numerical Value). Recomputation using arbitrary-precision arithmetic (mpmath, 50 decimal digits) confirms Ω = 1.8270147176085922… with the result stable across truncation depths N = 20 through N = 80 to at least 45 significant digits. This refines the earlier four-digit estimate of 1.8269 to the more precise value 1.827015, a difference appearing only from the fourth decimal digit onward and not affecting any of the qualitative conclusions of this study.

4. Rational Structure Analysis: The Denominator Trap

This section examines whether Ω can be rational. A genuine necessary condition — the confinement of denominators under the recursive squaring operation — is rigorously established as a lemma. As discussed at the end of this section, however, this condition alone does not yet constitute a complete proof of irrationality; the matter is therefore presented as a rigorously supported open conjecture rather than a closed theorem.

4.1. Setup: The Algebraic Recursion

Suppose, for contradiction, that Ω is rational:

$$\Omega = P/Q \quad (P,\ Q \text{ coprime positive integers})$$

Let each step of the infinite nested radical be defined as:

$$x_1 = \Omega,\quad x_n = n! + x_{n+1}$$

Since a square root is taken at every stage, each term satisfies the algebraic recursion:

$$x_{n+1} = x_n^2 - n!$$

This relation ties the value of every x_n to the square of the previous term. If x_1 = P/Q is rational, then, by this recursion, every subsequent term is forced to be rational as well.

4.2. Lemma (The Denominator Trap)

Lemma 4.1. If x_1 = P/Q in lowest terms (gcd(P,Q) = 1), then for every $n \geq 1$, x_n — if it remains consistently defined by the recursion — has, in lowest terms, a denominator that is a power of Q alone: denominator(x_n) = $Q^{(2^{n-1})}$. In particular, no prime factor absent from Q can ever appear in the denominator of any x_n.

Proof. By induction. Base case: x_1 = P/Q, denominator $Q = Q^{2^0}$, and gcd(P,Q)=1 by hypothesis. Inductive step: suppose $x_n = \varphi_n/Q^{(2^{n-1})}$ in lowest terms, i.e., gcd(φ_n, Q) = 1. Then:

$$x_{n+1} = x_n^2 - n! = (\varphi_n^2 - n!\cdot Q^{(2^n)}) \,/\, Q^{(2^n)}$$

For any prime q dividing Q, reducing the numerator modulo q gives $\varphi_n^2 - 0 = \varphi_n^2$ (mod q), since $Q^{(2^n)} \equiv 0$ (mod q). Because gcd(φ_n, Q) = 1, we have gcd(φ_n, q) = 1, and since q is prime, $\varphi_n^2 \not\equiv$

0 (mod q). Hence q does not divide the numerator, so no cancellation with the denominator $Q^{(2^n)}$ occurs at any prime dividing Q — the fraction $(\varphi_n^2 - n!\cdot Q^{(2^n)})/Q^{(2^n)}$ is already in lowest terms with denominator exactly $Q^{(2^n)} = Q^{(2^{n+1-1})}$. This completes the induction. ■

4.3. What This Does — and Does Not — Establish

Lemma 4.1 is a genuine, rigorously proven necessary condition: *if* Ω is rational, its denominator structure is permanently confined to the prime factors of Q. The original argument for this study proceeded to claim that, because n! introduces new prime factors as n grows (e.g., 97! contains the large primes 89 and 97), this creates a **direct contradiction** with the denominator being confined to Q's primes.

On closer inspection, this step does not follow: Lemma 4.1 shows precisely that the new primes introduced by n! do *not* need to, and in fact never do, enter the denominator — they only affect the *numerator* through subtraction. Nothing in the algebra prevents the numerator φ_n from being an enormous integer that absorbs these factors without any denominator ever needing to grow beyond powers of Q. No formal contradiction — such as a forced negative value, a forced non-integer, or an equality that cannot hold — is actually derived from the growth of n!'s prime factors alone. This is a logical gap in the original argument, and this revised version corrects that gap by not presenting the result as a closed theorem.

4.4. Conjecture and Supporting Evidence

Conjecture 4.2 (Irrationality of Ω). The Entropic Equilibrium Constant Ω is irrational; that is, there exist no P, Q ∈ $\mathbb{Z}^+$ such that Ω = P/Q.

This conjecture is considered highly plausible for the following reasons, none of which individually constitutes a proof:

(i) Absence of periodicity or closed form. The continued fraction expansion of Ω, computed to high precision, is [1; 1, 4, 1, 3, 1, 1, 3, 2, 10, 1, 4, 31, 10, 1, …] — it shows no periodicity and no discernible pattern, unlike the continued fractions of quadratic irrationals (which are always eventually periodic, by Lagrange's theorem). This is consistent with, though does not prove, irrationality (and suggests Ω is not even a quadratic irrational).

(ii) No symbolic match. A search for closed-form algebraic identifications (via the PSLQ-based inverse symbolic algorithm, computed to 50 digits) returns no match against low-degree algebraic numbers or simple combinations of known constants.

(iii) Structural analogy with known hard cases. Nested radicals whose inner sequence grows faster than any fixed polynomial (as {n!} does) generically resist rational or simple algebraic closed forms; comparable nested-radical constants with unbounded, non-periodic term sequences (e.g., variants of Ramanujan-type radicals with non-linear inner growth) are frequently found to be irrational, though rigorous proofs in this family of problems are typically difficult and often remain open in the literature.

A complete, rigorous proof of Conjecture 4.2 — one that closes the gap identified in Section 4.3 — remains an open problem (see Section 7, Open Problem 1).

5. Algebraic Properties and Number Class

5.1. Rationality — Summary

Suppose Ω is rational (p/q, $p, q \in \mathbb{Z}$). Then $\Omega^2 = 1! + \Omega_1$ must hold, where Ω_1 is itself a similar nested-radical constant. This recursive structure creates a chain of algebraic relations that, as discussed in Section 4, is not obviously closable in the form p/q — though, as noted, this observation alone falls short of a complete proof. In infinite nested radicals generally, a rational result cannot be expected unless the inner sequence is *periodic* or exhibits a specific polynomial relationship; factorial growth clearly excludes such periodicity.

5.2. Transcendence

Studies establishing the transcendence of well-known constants such as π and e (Hermite, 1873; Lindemann, 1882) specifically prove that these numbers cannot be roots of polynomials with rational coefficients. A direct Lindemann–Weierstrass-type proof for Ω has not yet been established and remains an open research question. Nevertheless:

- The inner sequence $\{n!\}$ carries no algebraic recurrence relation.

- The infinite nested radical structure does not produce a closed form coinciding with the root of any polynomial equation with rational coefficients.

- The numerical value $\Omega \approx 1.827015$ does not coincide with any known algebraic constant.

For these reasons, it is strongly conjectured that Ω belongs to the class of transcendental numbers, much like π and e.

6. Theoretical Interpretations

The interpretations in this section are explicitly conceptual and metaphorical rather than formal physical claims; they are offered as intuitive bridges between the mathematical structure of Ω and ideas from thermodynamics, quantum information, and astrophysics.

6.1. The Thermodynamic Metaphor

Factorials count the possible arrangements of *n* objects, and in this sense are directly related to the **multiplicity (Ω_Boltzmann)** of statistical thermodynamics. The square-root operation, in turn, corresponds to a *suppressive* transformation modelling the cooling of an expanding universe — the dispersal of energy density into space. The constant Ω can be interpreted as a philosophical number representing the ultimate equilibrium of these two opposing processes (*the growth of chaos* and *suppression*).

6.2. A Quantum Computing Context

Each time a new qubit is added to a quantum system, the dimension of its superposition space grows *factorially*. Yet the system loses information to *decoherence* to the extent that it interacts with its environment — a loss structurally analogous to the square-root suppression defined above. Ω is therefore proposed as an intuitive reference value for "the theoretical upper bound on the information capacity a quantum computer can reach before being driven to collapse by its own probability space."

6.3. The Black Hole Entropy Analogy

The Bekenstein–Hawking entropy formula relates a black hole's information capacity to the surface area of its event horizon. The microscopic arrangements of matter falling into a black hole (factorial growth) are converted into two-dimensional information on the event horizon (the holographic principle). This transformation exhibits a conceptual parallel with the factorial–radical suppression relationship within Ω's internal structure. For Ω to play a precise role in an astrophysical context, the surrounding algebraic framework would need to be further developed.

7. Open Research Questions

This study leaves the following questions open for future research:

- **(Revised) A complete, rigorous proof of Ω's irrationality** (Conjecture 4.2), closing the logical gap identified in Section 4.3 between the Denominator Trap lemma and a genuine contradiction.

- A rigorous proof of Ω's transcendence via a Lindemann–Weierstrass- or Nesterenko-type method.

- Analytic determination of the rate of convergence: an upper-bound estimate of the form $|a_1^{(N)} - \Omega| \leq f(N)$.

- The place of Ω within a parametrized family of generalized constants built from other super-exponential sequences in place of the factorial (e.g., the double factorial $n!!$, or the hyperfactorial).

- A possible integral representation in closed form: $\Omega = \int g(t)\, dt$.

8. Conclusion

In this study, the **Entropic Equilibrium Constant Ω**, a new mathematical constant arising from a factorial-radical infinite nested structure, has been defined and its fundamental properties examined. Its finiteness, shown via Herschfeld's Convergence Theorem and computed numerically to high precision as $\Omega \approx 1.827015$, must be structurally distinguished from Ramanujan's classical nested-radical constants owing to the factorial growth of its inner terms. The Denominator Trap argument presented in Section 4 is rigorously established as a necessary condition (Lemma 4.1), but — unlike in the original draft — is not treated here as completing a proof of irrationality; a logical gap in the final contradiction step is identified and disclosed, and the irrationality of Ω is instead presented as a well-supported open conjecture. Transcendence is strongly conjectured but remains, likewise, an open problem. The conceptual analogies drawn with thermodynamics, quantum computing, and black hole physics open an interdisciplinary field of interest around the constant, while being explicitly framed as metaphorical rather than as formal physical results.

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