Asymptotic Limit Behaviour of the Sum of Coprime Integers Generated by Euler's Totient Function

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Abstract

In number theory, the distribution of prime numbers and of integers coprime to a given integer exhibits highly irregular, seemingly chaotic behaviour term by term. This study builds a bridge between Discrete Mathematics (Number Theory) and Continuous Mathematics (Analysis) by examining the asymptotic limit behaviour of the sequence S_n, defined as the sum of all positive integers less than n that are coprime to n. The symmetry identity $S_n = n \cdot \varphi(n) / 2$ is first established; an algebraic squeeze inequality, derived from the natural bound $\varphi(n) \leq n - 1$, is then combined with the Squeeze (Sandwich) Theorem to prove that $\lim_{n\to\infty} (S_n^{1/n}) = 1$. The result shows that, despite the locally chaotic behaviour of Euler's totient function $\varphi(n)$, the n-th root limit of the sums it generates converges to a smooth, predictable constant (1).

Keywords

Euler's Totient Function; Squeeze (Sandwich) Theorem; Sequence Limits; Coprime Integers; Asymptotic Analysis.

Öz

Sayılar teorisinde asal sayıların ve aralarında asal tam sayıların dağılımı, dizisel olarak son derece düzensiz ve kaotik bir davranış sergiler. Bu çalışmada, n tam sayısından küçük ve n ile aralarında asal olan sayıların genel toplamını ifade eden S_n dizisinin sonsuzdaki asimptotik limit davranışı incelenmiştir. Öncelikle $S_n = n \cdot \varphi(n) / 2$ simetri özdeşliği ispatlanmış; ardından Euler'in φ fonksiyonunun doğal sınırından hareketle elde edilen cebirsel bir sıkıştırma eşitsizliği, Sıkıştırma Teoremi ile birlikte kullanılarak $\lim_{n\to\infty} (S_n^{1/n}) = 1$ olduğu ispatlanmıştır.

Anahtar Kelimeler

Euler φ Fonksiyonu; Sıkıştırma (Sandviç) Teoremi; Dizi Limitleri; Aralarında Asal Sayılar; Asimptotik Analiz.

1. Introduction and Motivation

In number theory, Euler's Totient Function ($\varphi(n)$) is used to determine the count of integers coprime to a given natural number n. By the very nature of this function, as the value of n increases, the values it produces exhibit sudden jumps and drops (depending on whether n is prime or composite); this irregularity has led $\varphi(n)$ to be presented as a typical example of a "chaotic arithmetic function" in classical number theory textbooks (Apostol, 1976; Niven et al., 1991).

Although the literature contains extensive studies on the asymptotic averages of $\varphi(n)$ (e.g., its limsup and liminf values, or the growth rate of the partial sum $\Sigma\varphi(k)$ on the order of n^2), the n-th root limit behaviour of the total sum of these numbers as $n \to \infty$ is generally not presented as a separate result in introductory analysis textbooks.

The primary motivation of this paper is to synthesize the number-theoretic approaches used in Olympiad problems examined by Aliyev and Çoker (2002) with the Squeeze (Sandwich) Theorem applications for sequences compiled by Uyhan (2002), in order to show how an algebraic sum that appears irregular converges, under sequential limit, to a smooth constant value (1). In this context, the study aims to build an interdisciplinary bridge by combining a classical number-theoretic identity (Proposition 2.1) with an analytic limit technique.

The remainder of the paper is organized as follows. Section 2 recalls basic definitions, the symmetry identity, and the Squeeze Theorem. Section 3 proves the main theorem. Section 4 discusses the findings and addresses the study's limitations and possible generalizations.

2. Basic Concepts and Preliminary Theorems

Definition 2.1. For an integer $n \geq 2$, let C_n denote the set of all positive integers less than n that are coprime to n (i.e., $\gcd(k, n) = 1$). The cardinality of this set is called Euler's totient function and is defined as:

$$|C_n| = \varphi(n)$$

Definition 2.2. Let S_n denote the sum of all elements in the set C_n:

$$S_n = \Sigma_{1 \leq k < n,\ gcd(k,n)=1}\ k$$

Proposition 2.1 (Symmetry Property). The sum S_n is algebraically equivalent to the following expression:

$$S_n = n \cdot \varphi(n) / 2$$

Proof. It is a basic property of the Euclidean algorithm that if $\gcd(k, n) = 1$, then necessarily $\gcd(n - k, n) = 1$. This shows that coprime integers form symmetric pairs of the form k and $(n - k)$. Grouping the elements of C_n into such pairs, each pair sums to $k + (n - k) = n$. Since the set contains a total of $\varphi(n)$ elements, there are exactly $\varphi(n)/2$ such pairs. The total sum is therefore $S_n = n \cdot \varphi(n)/2$. ■

Remark 2.1. For small values such as $n = 1$ or $n = 2$, $\varphi(n) = 1$, and the pairing argument degenerates (the case $k = n - k$); however, this does not affect the asymptotic behaviour as $n \to$

∞, which is the focus of this study, since the asymptotic result of interest is meaningful only for sufficiently large n.

Theorem 2.1 (Squeeze/Sandwich Theorem). Let (a_n), (b_n), (c_n) be real sequences such that $a_n \leq b_n \leq c_n$ for all $n \geq N$, and suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n = L$ (Uyhan, 2002).

Proposition 2.2 (Fundamental Limits). As proven using Bernoulli's inequality in continuous analysis (Uyhan, 2002), for a positive constant c, the following sequence limits hold:

$$1.\ \lim_{n\to\infty} n^{1/n} = 1 \qquad 2.\ \lim_{n\to\infty} c^{1/n} = 1$$

3. Main Theorem and Proof

Theorem 3.1 (Asymptotic Limit of the Euler Sum). For $n \geq 2$, the limit of the n-th root of the sequence S_n, the total sum of integers coprime to n, as $n \to \infty$ is 1:

$$\lim_{n\to\infty} (S_n^{1/n}) = 1$$

Proof.

1. First, let us establish the natural bounds of $\varphi(n)$. For $n \geq 2$, $\varphi(n)$ can take a minimum value of 1. It attains its maximum value when n is prime, in which case $\varphi(n) = n - 1$. Hence the following strict inequality holds:

$$1 \leq \varphi(n) \leq n - 1 < n$$

2. To construct the identity $S_n = n \cdot \varphi(n)/2$ obtained in Proposition 2.1, let us multiply every term of the above inequality by $n/2$:

$$n/2 \leq n \cdot \varphi(n)/2 < n^2/2$$

Rewriting this in terms of S_n:

$$n/2 \leq S_n < n^2/2$$

3. To obtain the limit value, let us take the n-th root (the $1/n$ power) of every term of the inequality:

$$(n/2)^{1/n} \leq S_n^{1/n} < (n^2/2)^{1/n}$$

4. Let us now compute the limits of the left- and right-hand sides as $n \to \infty$, based on the rules given in Proposition 2.2.

Limit of the lower bound (left-hand side):

$$\lim_{n\to\infty} (n/2)^{1/n} = \lim n^{1/n} / \lim 2^{1/n} = 1/1 = 1$$

Limit of the upper bound (right-hand side):

$$\lim_{n\to\infty} (n^2/2)^{1/n} = (\lim n^{1/n})^2 / \lim 2^{1/n} = 1^2/1 = 1$$

5. As stated in Theorem 2.1 (the Squeeze Theorem), the lower and upper bounds of the sequence $S_n^{1/n}$ converge to the same value (1) in the limit. This forces the squeezed sequence to also converge to the same limit. Therefore:

$$\lim_{n\to\infty} (S_n^{1/n}) = L = 1 \ \blacksquare$$

4. Conclusion, Discussion, and Limitations

The theorem proven in this paper clearly demonstrates how the seemingly irregular elements of number theory (coprime integers) submit to the rules of macro-analysis when viewed from the perspective of limits at infinity. Although the graph of the function $\varphi(n)$ exhibits an extremely chaotic zigzag pattern, the geometric-like growth characteristic of the sum of these numbers (S_n) is smoothed out by the rigorous analytic structure of the Squeeze Theorem and settles into a smooth constant, $1^{1/n}$. This result is an elegant example of the smoothing power of the concept of a limit.

The scope of this result is limited to the n-th root limit (the exponential order of growth); the sequence S_n itself grows without bound (on the order of S_n *n*²/2), and the proven result does not change the fact that *S*n~ diverges — it only shows convergence to 1 on the exponential/root scale. Furthermore, deriving sharper asymptotic expressions (e.g., average-order results) using the exact value of $\varphi(n)$, rather than only its bounds, is left outside the scope of this study.

Future research may address the following questions:

1. How can the exact asymptotic growth order of S_n be expressed more sharply, in relation to known average-order results such as $\Sigma\varphi(k) \sim 3n^2/\pi^2$?

2. Can a similar squeeze technique be applied to sum sequences defined using other multiplicative arithmetic functions (e.g., the divisor-counting function $d(n)$ or the divisor-sum function $\sigma(n)$) instead of $\varphi(n)$?

Ethics Statement and Additional Information

Ethics Committee Approval: This study is a theoretical/analytical mathematics investigation that does not involve any experimentation on human or animal subjects; therefore, ethics committee approval is not required.

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