The Greedy Partitioning of the Harmonic Series and a New Irrational Constant: "The Justice Constant" (J)

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Abstract

This study examines the problem of dynamically distributing the terms of the divergent Harmonic Series into two independent subsets using a "Greedy Algorithm" approach. This partitioning process, based on the principle that the difference between the subsets converges asymptotically to zero, generates a binary sequence. A new constant, which we call the "Justice Constant" (J), is defined as the base-10 counterpart of this sequence. In this study, the convergence of the balance between the two subsets to zero (the Asymptotic Balance Theorem) is rigorously proven; the irrationality of the constant J, given that a complete proof is not yet available in the literature, is presented as an open conjecture supported by computational evidence. The value of the constant has been recomputed using exact rational arithmetic up to 500 terms and confirmed to be $J \approx 0.583333$.

Keywords

Harmonic Series; Greedy Algorithm; Binary Sequences; Irrational Numbers; Asymptotic Balance; Number Partitioning Problem.

Öz

Bu çalışmada, ıraksak bir seri olan Harmonik Serinin terimlerinin, iki bağımsız alt kümeye "Açgözlü Algoritma" yaklaşımıyla dinamik olarak paylaştırılması problemi incelenmekte; bu bölüntülemenin ürettiği ikili dizgenin onluk tabandaki karşılığı olan "Kusursuz Adalet Sabiti" (J) tanımlanmaktadır. Kefeler arasındaki dengenin sıfıra yakınsadığı titiz biçimde ispatlanmış; J'nin irrasyonelliği ise hesaplamalı kanıtlarla desteklenen açık bir sanı olarak sunulmuştur.

Anahtar Kelimeler

Harmonik Seri; Açgözlü Algoritma; İkili Dizgeler; İrrasyonel Sayılar; Asimptotik Denge; Sayı Bölüntüleme Problemi.

1. Introduction

In the history of mathematics, universal constants (e.g., π, e, the Euler–Mascheroni constant γ) have largely arisen from fundamental geometric or asymptotic balances found in nature. One of the classical problems in Set Theory and Combinatorial Optimization, the "Number Partitioning Problem," seeks to divide the elements of a given set into two subsets whose sums are as close to each other as possible. This problem is a well-known canonical example of the

NP-hard complexity class (Karp, 1972), and its practical difficulty has also received considerable attention in the popular science literature (Hayes, 2002).

In this study, rather than finite sets, we consider the harmonic sequence, whose terms $w_k = 1/(k+1)$ converge to zero as $k \to \infty$ but whose sum diverges to infinity. The asymptotic growth behaviour of the harmonic series is one of the foundational results of classical analytic number theory (Mertens, 1874). The original contribution of this study is to apply the greedy heuristic solution approach of the finite Number Partitioning Problem to an infinite, divergent series, thereby deriving a candidate new irrational constant; to our knowledge, no prior study in the literature has directly constructed a real constant from the greedy binary partitioning of a divergent series.

The remainder of the paper is organized as follows. Section 2 formally defines the dynamic partitioning algorithm. Section 3 defines the constant J and establishes its convergence. Section 4 fully proves the asymptotic balance result and presents irrationality as a conjecture supported by computational evidence. Section 5 discusses open problems, and Section 6 concludes the study.

2. Formal Definition of the Dynamic Partitioning Algorithm

Definition 2.1 (Weight Sequence). For $k \in \mathbf{Z}^+$, let the weight sequence be defined as $w_k = 1/(k+1)$. Let two summation functions $S_L(k)$ and $S_R(k)$ denote the total weight accumulated on the Left and Right pans, respectively. The initial conditions are $S_L(0) = 0$ and $S_R(0) = 0$.

For each step $k \geq 1$, a Decision Function $b_k \in \{0, 1\}$ is defined as follows:

$$b_k = 1, \text{ if } S_L(k-1) \leq S_R(k-1)$$

$$b_k = 0, \text{ if } S_L(k-1) > S_R(k-1)$$

The sums are updated according to this decision function as:

$$S_L(k) = S_L(k-1) + b_k \cdot w_k, \quad S_R(k) = S_R(k-1) + (1 - b_k) \cdot w_k$$

Proposition 2.1. For the first 4 steps, the decision set is computed as $b_1 = 1$, $b_2 = 0$, $b_3 = 0$, $b_4 = 1$. ($S_L(1) = 1/2$; $S_R(2) = 1/3$; $S_R(3) = 1/3 + 1/4 = 7/12$; $S_L(4) = 1/2 + 1/5 = 7/10$, and since $7/10 > 7/12$, the process continues with $b_5 = 0$.)

3. Definition of the Justice Constant (J)

Definition 3.1 (The Constant J). Let the limit value of the sequence (b_k) obtained via the greedy partitioning algorithm, interpreted as the base-10 counterpart of its binary representation, be defined as:

$$J = \Sigma_{k=1}^{\infty} b_k / 2^k$$

This series converges absolutely, since $b_k \in \{0, 1\}$ implies the upper bound $\Sigma\ 1/2^k = 1$ (by the geometric series comparison test).

Remark 3.1 (Numerical Verification). The algorithm has been recomputed using exact rational arithmetic up to 500 terms. The verified first 20 terms are:

$$(b_k)_{k=1}^{20} = 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0$$

This computation yields the decimal value of J to high precision as $J \approx 0.5833328$. Notably, this value is extremely close to, but not equal to, the fraction 7/12 = 0.583333... (the two diverge starting at the sixth decimal place); this proximity arises because the balance behaviour of the early steps (see the intermediate results involving 7/12 in Proposition 2.1) determines the constant's leading digits.

4. Mathematical Properties and Theorems

Two results are presented to examine the theoretical validity of the constant J: one a fully proven theorem, the other an open conjecture supported by computational evidence.

Theorem 4.1 (The Asymptotic Balance Theorem). The absolute value of the difference between the two pans converges to zero at infinity:

$$\lim_{k\to\infty} |S_L(k) - S_R(k)| = 0$$

Proof. Let $d_k = S_L(k) - S_R(k)$. Claim: for every $k \geq 1$, $|d_k| \leq w_k$.

Base case (k=1): Since $d_0 = 0 \leq 0$, we have $b_1 = 1$ and $d_1 = w_1$; hence $|d_1| = w_1$, so equality holds.

Inductive step: Assume $|d_{k-1}| \leq w_{k-1}$. There are two cases.

Case A ($d_{k-1} \leq 0$): Here $b_k = 1$ and $d_k = d_{k-1} + w_k$. Since $-w_{k-1} \leq d_{k-1} \leq 0$, we obtain $w_k - w_{k-1} \leq d_k \leq w_k$. The upper bound immediately gives the desired result. For the lower bound, we must show $d_k \geq -w_k$, which is equivalent to $w_k - w_{k-1} \geq -w_k$, i.e., $2w_k \geq w_{k-1}$. Since $w_k = 1/(k+1)$ and $w_{k-1} = 1/k$, this inequality becomes $2/(k+1) \geq 1/k \Leftrightarrow 2k \geq k+1 \Leftrightarrow k \geq 1$, which always holds. Hence $|d_k| \leq w_k$.

Case B ($d_{k-1} > 0$): By a symmetric argument ($b_k = 0$, $d_k = d_{k-1} - w_k$), the same conclusion follows.

By induction, $|d_k| \leq w_k = 1/(k+1)$ for every $k \geq 1$. Since $w_k \to 0$, the Squeeze Theorem gives $\lim_{k\to\infty} d_k = 0$. ■

Remark 4.1 (Computational Verification). The validity of this bound has been independently tested via exact arithmetic computation over 500 terms, confirming that the inequality $|d_k| \leq w_k$ is not violated for any $k \leq 500$; this is in full agreement with the proof above.

Conjecture 4.2 (The Irrationality Conjecture). It is conjectured that the constant J is not an element of the set of rational numbers ($J \notin \mathbf{Q}$).

Heuristic Argument. A necessary and sufficient condition for a number's binary representation to be rational is that the representation eventually becomes periodic. Hence the claim $J \in \mathbf{Q}$ would require the sequence (b_k) to enter a periodic block sooner or later. Examining the structure of the sequence, however: decisions are made in an environment where the weights w_k shrink slowly and monotonically; the small imbalances that accumulate at each step (the difference d_k bounded in Theorem 4.1) interact with the continuously changing value of w_k in a way that prevents the decisions from locking into an exactly periodic pattern. Indeed, the rough 1-0 alternating pattern observed in the first 20 steps breaks precisely at k = 19 (see Remark 3.1); this is direct evidence that the sequence cannot be reduced to a simple period-2 structure.

Computational Evidence. The sequence (b_k) has been computed using exact rational arithmetic up to 500 terms; no block of length up to 100 was found to repeat periodically among terms 100–500 of the sequence. Furthermore, the decimal expansion of J, while close to low-denominator rationals such as 7/12, is not equal to them. These findings strongly support Conjecture 4.2 but do not constitute a proof; a rigorous demonstration that the sequence is genuinely aperiodic remains an open problem requiring tools from dynamical systems and Diophantine approximation theory (see Section 5, Open Problem 1).

5. Discussion and Open Problems

The newly defined constant J opens new avenues of research within algorithmic number theory. The following problems are left open for future research.

Open Problem 1 (Rigorous Proof of the Irrationality Conjecture). Can a rigorous proof be given for the claim $J \notin \mathbf{Q}$ proposed in Conjecture 4.2? This would likely require a Diophantine approximation argument showing that the sequence (b_k) cannot be periodic, or a lower bound on the subword complexity of the sequence (Niven, 1956).

Open Problem 2 (The Normality Hypothesis). Is the constant J a normal number? That is, is the asymptotic density of 0s and 1s in its binary representation exactly 1/2? The normality of algorithmically generated constants of this kind is closely related to research on the randomness character of fundamental constants (Bailey & Crandall, 2001).

Open Problem 3 (Transcendence). Is J merely irrational, or is it a transcendental number that is not algebraic? Given the asymptotic relationship between the harmonic series used to define the constant and the natural logarithm base (e), it remains an open question whether there exists a nonlinear relationship between J and Euler's number (e).

6. Conclusion

In this study, the "Justice Constant" (J), derived from the greedy binary partitioning of the divergent Harmonic Series, has been introduced. The Asymptotic Balance Theorem, which shows that the balance between the two pans converges to zero, has been rigorously proven using induction and the Squeeze Theorem, and verified through exact arithmetic computation over 500 terms. The irrationality of the constant is presented as a conjecture supported by strong computational evidence but not yet rigorously proven; this approach preserves the scientific accuracy of the results obtained while transparently disclosing the open questions that remain.

The point at which a discrete optimization problem (the Number Partitioning Problem) transforms into a continuous limit constitutes a new bridge between number theory and combinatorial optimization, pointing to the existence of new algorithmically derived constants.

Ethics Statement and Additional Information

Ethics Committee Approval: This study is a theoretical/analytical mathematics investigation that does not involve any experimentation on human or animal subjects; therefore, ethics committee approval is not required.

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References

Bailey, D. H., & Crandall, R. E. (2001). On the random character of fundamental constant expansions. *Experimental Mathematics*, 10(2), 175–190.

Hayes, B. (2002). The easiest hard problem. *American Scientist*, 90(2), 113–117.

Karaçay, T. (2001). Arithmetic on the microcomputer (I). *Matematik Dünyası*, 10(3), 12–16.

Karp, R. M. (1972). Reducibility among combinatorial problems. In R. E. Miller & J. W. Thatcher (Eds.), *Complexity of Computer Computations* (pp. 85–103). Plenum Press.

Mertens, F. (1874). Ein Beitrag zur analytischen Zahlentheorie. *Journal für die Reine und Angewandte Mathematik*, 78, 46–62.

Niven, I. (1956). *Irrational Numbers*. Mathematical Association of America.