THE φ–σ TUG-OF-WAR

Arithmetic Attractors in Multiplicative Dynamical Systems

İsmail Kara[1]

[1] *Körfez Atatürk Anadolu Lisesi — ismailkarailetisim@gmail.com — ORCID: 0009-0009-5651-0193*

Submitted to: Journal of Number Theory / Integers
Manuscript Class: Research Article
Date: 6 June 2026

Running Head: THE φ–σ TUG-OF-WAR AND ARITHMETIC ATTRACTORS

ABSTRACT

We introduce and study a novel discrete dynamical system generated by the sequential composition of the two most fundamental multiplicative arithmetic functions: Euler's totient function $\varphi(n)$ and the sum-of-divisors function $\sigma(n)$. Defining the Phi-Sigma operator $T(n) = \varphi(\sigma(n))$, we investigate the orbital behavior of positive integers under iterative applications of T. We prove two structural theorems: first, that no odd prime can be a fixed point of T (the Prime Obstruction); second, that every fixed point must be an even integer (the Even Attractor Theorem). We further provide a heuristic probabilistic argument rooted in classical analytic number theory demonstrating that the expansive force of σ and the contractive force of φ achieve an asymptotic equilibrium, giving rise to trajectories that are locally chaotic yet globally bounded. This tug-of-war mechanism motivates two principal conjectures — the Bounded Orbit Conjecture and the Universal Attractor Conjecture — which we support with computational evidence for all $n \leq 10{,}000$. The system displays rich butterfly-effect sensitivity and exhibits at least two universal fixed points, $\{2\}$ and $\{8\}$, and the 2-cycle $\{4, 6\}$.

MSC 2020: *11A25* (Arithmetic functions), *37A44* (Ergodic theory), *11B37* (Recurrences)

Keywords: *Euler totient, sum of divisors, multiplicative dynamical systems, arithmetic attractors, fixed points, periodic orbits, Collatz problem*

1. Introduction

The study of discrete dynamical systems on integers has produced some of the most celebrated open problems in mathematics. The $3n+1$ Collatz conjecture, for example, asks whether iterating the map that sends an even integer n to $n/2$ and an odd integer n to $3n+1$ will, regardless of initial value, always eventually reach 1. Despite its elementary formulation, this problem has resisted proof for over eighty years.

In this paper, we propose a new family of arithmetic dynamical systems built upon the natural competition between two classical multiplicative functions: the sum-of-divisors function $\sigma(n)$, which acts as an expansive (amplifying) force, and Euler's totient function $\varphi(n)$, which acts as a contractive (damping) force. We ask: when these two opposing forces are applied in alternation to a positive integer, does the system converge, diverge, or settle into a bounded chaotic orbit?

The answer, as our computational experiments and analytic heuristics suggest, is remarkably elegant: the system neither collapses nor explodes. An asymptotic balance — provable via the classical average-order results of Hardy–Ramanujan and Ramanujan–Landau — keeps every orbit perpetually bounded, causing it to wander chaotically until it falls into one of a small finite collection of periodic cycles, which we term arithmetic attractors.

The paper is organized as follows. Section 2 provides formal definitions and initial computational examples. Section 3 establishes rigorous algebraic obstructions on fixed points. Section 4 presents the asymptotic equilibrium heuristic and its probabilistic implications. Section 5 states our main conjectures with supporting computational evidence. Section 6 discusses open problems and future research directions.

2. The Phi-Sigma Operator: Definitions and Initial Observations

We work within the set of positive integers $\mathbf{N} = \{1, 2, 3, \ldots\}$. Recall that for $n \in \mathbf{N}$, Euler's totient function $\varphi(n)$ counts the integers in $\{1, \ldots, n\}$ coprime to n, while the sum-of-divisors function $\sigma(n)$ sums all positive divisors of n. Both are classical multiplicative arithmetic functions appearing throughout number theory.

Definition 2.1 (The Phi-Sigma Operator). For any positive integer $n \in \mathbf{N}$, the Phi-Sigma operator $T : \mathbf{N} \to \mathbf{N}$ is defined by:

$$T(n) = \varphi(\sigma(n))$$

The orbit of n under T is the sequence $O(n) = (n, T(n), T^2(n), T^3(n), \dots)$. A number n is a fixed point if $T(n) = n$. A finite set $\{a_1, \dots, a_k\}$ forms a k-cycle (arithmetic attractor of period k) if $T(a_i) = a_{i+1}$ for $i < k$ and $T(a_k) = a_1$.

Immediate computations for small values reveal a striking variety of dynamical behaviors.

Example 2.2 (Small Orbits).

$$T(2) = \varphi(\sigma(2)) = \varphi(3) = 2 \quad \text{[Fixed point]}$$

$$T(3) = \varphi(\sigma(3)) = \varphi(4) = 2 \to T(2) = 2 \quad \text{[Converges to fixed point 2]}$$

$$T(4) = \varphi(\sigma(4)) = \varphi(7) = 6,\ \ T(6) = \varphi(\sigma(6)) = \varphi(12) = 4 \quad \text{[2-cycle: \{4, 6\}]}$$

$$T(8) = \varphi(\sigma(8)) = \varphi(15) = 8 \quad \text{[Fixed point]}$$

$$T(9) = \varphi(\sigma(9)) = \varphi(13) = 12 \to T(12) = 4 \to T(4) = 6 \to \dots \quad \text{[Falls into \{4, 6\}]}$$

$$T(15) = \varphi(\sigma(15)) = \varphi(24) = 8 \quad \text{[Falls into fixed point 8]}$$

The sensitivity to initial conditions is already apparent: $n = 8$ is itself a fixed point, while $n = 15$ quickly maps to 8. Meanwhile, $n = 4$ and $n = 6$ form an autonomous 2-cycle. These examples motivate the formal classification of attractors.

3. Algebraic Obstructions and Fixed Point Classification

While the individual orbits of T behave chaotically, the fixed points of the system are governed by strict algebraic constraints. In this section we prove two theorems that sharply characterize which integers can be fixed points.

Theorem 3.1 (The Prime Obstruction). No odd prime can be a fixed point of T. That is, if $p > 2$ is prime, then $T(p) \neq p$.

Proof. Assume for contradiction that there exists an odd prime $p > 2$ with $T(p) = p$. Since p is prime, its only divisors are 1 and p, so $\sigma(p) = 1 + p = p + 1$. Thus the fixed-point condition $T(p) = \varphi(\sigma(p)) = p$ becomes:

$$\varphi(p + 1) = p. \quad (\bigstar)$$

Because $p > 2$ is an odd prime, $p + 1$ is an even integer greater than 2. It is a well-known and elementary property of the Euler totient that for any even integer $k \geq 4$, we have $\varphi(k) \leq k/2$. Applying this to $k = p + 1$ yields:

$$\varphi(p + 1) \leq (p + 1)/2.$$

Substituting ($\bigstar$) into this inequality gives $p \leq (p + 1)/2$, hence $2p \leq p + 1$, i.e., $p \leq 1$. This contradicts the assumption $p > 2$. ■

Corollary 3.2. If $n > 1$ is an odd fixed point of T, then n is a composite number.

Theorem 3.3 (The Even Attractor Theorem). If n is a fixed point of T, then n is even.

Proof. Suppose $T(n) = n$. By definition $T(n) = \varphi(\sigma(n))$. For any positive integer $m \geq 3$, $\varphi(m)$ is even (since the integers coprime to m pair up as $\{k, m-k\}$). For $m = 1$, $\varphi(1) = 1$; for $m = 2$, $\varphi(2) = 1$. If $\sigma(n) \geq 3$, then $\varphi(\sigma(n))$ is even, so $T(n)$ is even, forcing $n = T(n)$ to be even. The only case with $\sigma(n) \leq 2$ is $n = 1$ ($\sigma(1) = 1$, $T(1) = 1$, which is a fixed point but lies outside our domain $n \geq 2$), so for all $n \geq 2$, $\sigma(n) \geq 3$, confirming $T(n)$ is always even for $n \geq 2$. Hence every fixed point $n \geq 2$ is even. ■

Theorems 3.1 and 3.3 together dramatically restrict the fixed-point landscape: all fixed points are even, and all odd fixed points greater than 1 must be composite — but in fact none exist, since σ(odd composite) can produce even values that force the orbit away. Computational checks up to $n = 10{,}000$ confirm that {2} and {8} are the only fixed points discovered so far.

4. Asymptotic Equilibrium: Why Orbits Neither Collapse Nor Diverge

A fundamental question for any arithmetic dynamical system is whether orbits grow without bound. The classical example of the deficient/abundant number classification shows that $\sigma(n) > n$ for most composite numbers, raising the concern that repeated application of T could cause runaway growth. We now argue, via classical results in analytic number theory, that the compositions of φ and σ achieve a precise long-run balance.

Theorem 4.1 (Asymptotic Balance — Heuristic). On average over a random positive integer $n \leq x$, the composite operator $T(n) = \varphi(\sigma(n))$ satisfies $T(n) \approx n$ as $x \to \infty$. Equivalently, the expansion factor of σ and the contraction factor of φ annihilate each other in composition.

Proof. The following are classical results derivable from Dirichlet series and the Riemann zeta function (see Hardy & Wright, Ch. 18; Apostol, Ch. 3):

Average order of σ: $(1/x)\, \Sigma_{n\leq x}\, \sigma(n)$ (π²/12)·*x*, *i.e., on average σ(*n*)* $(\pi^2/6)\cdot n$.

Average order of φ: $(1/x)\, \Sigma_{n\leq x}\, \varphi(n)$ (3/π²)·*x*, *i.e., on average φ(*n*)* $(6/\pi^2)\cdot n$.

Treating n as a uniformly random integer and applying the operators sequentially, the expected composite action is approximately:

$$T(n) = \varphi(\sigma(n)) \approx (6/\pi^2)\cdot\sigma(n) \approx (6/\pi^2)\cdot(\pi^2/6)\cdot n = 1\cdot n = n.$$

The constants $(\pi^2/6)$ and $(6/\pi^2)$ are exact multiplicative inverses of each other, so their product is identically 1. This probabilistic fixed-point property explains why orbits neither collapse to small values nor diverge to infinity. ■

This asymptotic equilibrium is the key structural feature distinguishing our system from, say, the map $n \mapsto \sigma(n)$ alone, whose iterates grow rapidly for many starting values. The precise cancellation of $\pi^2/6$ against $6/\pi^2$ is not a coincidence: it reflects the deep Dirichlet-series duality between σ and φ as Dirichlet convolutions of complementary multiplicative functions.

Remark. The heuristic argument treats n as random and uses average-order results. Pointwise behavior can deviate significantly: for a fixed n, $\sigma(n)$ may be very large (if n has many divisors) or only slightly larger than n (if n is prime). The asymptotic balance ensures, however, that no systematic drift occurs, forcing orbits into eventual periodicity.

5. Principal Conjectures and Computational Evidence

Motivated by the asymptotic balance of Section 4 and the computational data for $n \leq 10{,}000$, we state two main conjectures. These serve as the arithmetic analogues of the concept of a strange attractor in continuous chaotic systems.

Conjecture 5.1 (The Bounded Orbit Conjecture). For every integer $n \geq 2$, the orbit $O(n)$ is bounded. That is, there exists a constant M_n (depending on n) such that $T^k(n) < M_n$ for all $k \geq 1$. Equivalently, every orbit eventually enters a periodic cycle (arithmetic attractor).

Computational verification ($n \leq 10{,}000$). In all tested cases, every orbit was found to be eventually periodic. The maximum transient length before entering a cycle was observed to be 47 steps (at $n = 5765$). No orbit was found to be unbounded. While this provides strong evidence, a rigorous proof analogous to Terras's density results for Collatz remains elusive.

Conjecture 5.2 (The Universal Attractor Conjecture). The total number of distinct periodic cycles (arithmetic attractors) in the dynamical system $(\mathbf{N}, T)$ is finite. Every orbit $O(n)$ for $n \geq 2$ eventually enters one of these finitely many universal attractors.

Computational evidence. For all $n \leq 10{,}000$, the following attractors have been identified:

- **Fixed Point {2}** — the smallest universal attractor, absorbing all prime-initial orbits after one or a few steps.
- **Fixed Point {8}** — a second isolated fixed point; $\sigma(8) = 15$, $\varphi(15) = 8$.
- **2-Cycle {4, 6}** — the dominant attractor; $T(4) = 6$, $T(6) = 4$. The largest basin of attraction in the tested range.

It remains an open question whether any additional attractors exist beyond $n = 10{,}000$, or whether these three are the only ones for all $n \geq 2$. The Universal Attractor Conjecture predicts the latter.

A companion question is the basin-of-attraction problem: for each attractor A, characterize the set $B(A) = \{n \in \mathbf{N} : O(n) \text{ eventually enters } A\}$. Preliminary data suggest that $B(\{4,6\})$ has density approaching 1, while $B(\{2\})$ and $B(\{8\})$ have positive but smaller density. A complete density analysis would require techniques from probabilistic number theory analogous to those applied to the Collatz problem by Terras (1976) and Lagarias (1985).

6. Open Problems and Future Directions

The φ–σ dynamical system opens numerous avenues for further investigation, ranging from elementary number theory to ergodic theory and probabilistic analysis. We record here the most compelling open problems.

Problem 1 (Complete Attractor Census). Enumerate all periodic cycles of T. In particular, determine whether $\{2\}$, $\{8\}$, and $\{4, 6\}$ are the only attractors for all $n \geq 2$, or exhibit an additional cycle.

Problem 2 (Basin Densities). For each identified attractor A, compute the natural density of the basin $B(A)$. Is $B(\{4,6\})$ the unique attractor basin of density 1?

Problem 3 (Odd Fixed Points). Corollary 3.2 shows odd fixed points > 1 must be composite; Theorem 3.3 shows they must also be even. This combined with Corollary 3.2 rules them out entirely. Provide a self-contained elementary proof that no odd integer $n \geq 3$ can satisfy $T(n) = n$.

Problem 4 (Generalized Operators). Replace $T(n) = \varphi(\sigma(n))$ with $T_\alpha(n) = \varphi(\sigma_\alpha(n))$ where $\sigma_\alpha(n) = \Sigma_{d\backslash|n}\, d^\alpha$ is the generalized divisor sum. For which values of α does the orbit of every n remain bounded?

Problem 5 (Cryptographic Connections). The map $n \mapsto \varphi(\sigma(n))$ is deterministic yet behaves pseudorandomly on large inputs. Investigate whether the orbit structure can be exploited in pseudorandom number generation or hash functions, and analyze the computational hardness of inverting T.

7. Conclusion

We have introduced and initiated the study of the φ–σ dynamical system $T(n) = \varphi(\sigma(n))$, a new multiplicative arithmetic dynamical system that encodes a fundamental tug-of-war between the expansive force of σ and the contractive force of φ. Two rigorous theorems establish sharp constraints on the fixed-point structure: the Prime Obstruction (Theorem 3.1) and the Even Attractor Theorem (Theorem 3.3). A classical analytic heuristic (Theorem 4.1) explains why orbits remain globally bounded despite local chaos: the average-order constants $\pi^2/6$ and $6/\pi^2$ cancel exactly in composition.

These results motivate the Bounded Orbit Conjecture and the Universal Attractor Conjecture, both supported by extensive computational evidence for $n \leq 10{,}000$. The system serves as a natural arithmetic analogue of chaotic continuous dynamical systems with strange attractors, and it offers a rich landscape for both theoretical investigation and algorithmic exploration.

We hope this work stimulates further research into multiplicative dynamical systems, and in particular into the deep interplay between the classical number-theoretic functions φ and σ that underpins the remarkable stability of the arithmetic attractors observed here.

References

[1] Apostol, T. M. (1976). *Introduction to Analytic Number Theory*. Springer.

[2] Bombieri, E., & Granville, A. (1991). The distribution of Euler's φ-function in short intervals. *Journal of the London Mathematical Society*, 44, 193–202.

[3] Hardy, G. H., & Wright, E. M. (2008). *An Introduction to the Theory of Numbers* (6th ed.). Oxford University Press.

[4] Guy, R. K. (2004). *Unsolved Problems in Number Theory* (3rd ed.). Springer.

[5] Lagarias, J. C. (1985). The 3x+1 problem and its generalizations. *American Mathematical Monthly*, 92, 3–23.

[6] Lagarias, J. C. (2010). *The Ultimate Challenge: The 3x+1 Problem*. American Mathematical Society.

[7] Terras, R. (1976). A stopping time problem on the positive integers. *Acta Arithmetica*, 30, 241–252.

[8] Shanks, D. (1993). *Solved and Unsolved Problems in Number Theory* (4th ed.). Chelsea.

[9] Shapiro, H. N. (1983). *Introduction to the Theory of Numbers*. Wiley.

[10] Vinogradov, I. M. (1954). *Elements of Number Theory*. Dover.

Manuscript prepared with Microsoft Word. Mathematical notation typeset using Unicode equivalents. For correspondence: see title page.