A Topological Study on Quadratic Prime Graphs and the Generalized 4-Difference Impossibility Theorem

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Abstract

This study brings together prime numbers—one of the central objects of number theory—and the topological modelling capacity of graph theory to define a new algebraic-combinatorial structure called the "Quadratic Prime Graph" (G_Q), in which two primes are joined by an edge if and only if their sum is a perfect square. The main objective is to examine the connectivity properties of this structure and, in particular, to show that prime pairs differing by 4 (e.g., 3 and 7) can never share a common neighbour in this graph. The proof relies on a system of Diophantine equations and proof by contradiction; the result is then generalized to cover all "cousin prime" pairs (the Generalized 4-Difference Impossibility Theorem). The findings show, on purely algebraic grounds, that certain substructures of G_Q are triangle-free, and that number-theoretic graphs of this kind can be fruitfully studied with classical graph-theoretic tools.

Keywords

Prime Numbers; Graph Theory; Diophantine Equations; Topological Number Theory; Triangle-free Graphs; Cousin Primes.

Öz

Bu çalışmada, sayılar teorisinin temel araştırma nesnesi olan asal sayılar ile çizge kuramının (graph theory) sunduğu topolojik modelleme araçları bir araya getirilerek, "Karesel Asal Çizgesi" (Quadratic Prime Graph, G_Q) adı verilen yeni bir cebirsel-kombinatoryal yapı tanımlanmaktadır. Bu çizgede iki asal sayı, ancak ve ancak toplamları bir tam kare oluşturduğunda bir ayrıtla birleştirilmektedir. Çalışmanın temel amacı, bu yapının bağlantısallık özelliklerini incelemek ve özellikle aralarındaki fark 4 olan asal sayı çiftlerinin bu çizgede hiçbir zaman ortak bir komşuya sahip olamayacağını göstermektir. İspat, Diophantine denklem sistemleri ve çelişki yöntemi kullanılarak elde edilmiş; sonuç, tüm "kuzen asal" çiftlerini kapsayacak biçimde genelleştirilmiştir.

Anahtar Kelimeler

Asal Sayılar; Çizge Kuramı; Diophantine Denklemleri; Topolojik Sayılar Teorisi; Üçgensiz Çizgeler; Kuzen Asallar.

1. Introduction

In the number theory literature, research on the additive properties of prime numbers has generally focused on notoriously difficult problems such as the Goldbach Conjecture. On the other hand, ever since Euclid, the relationships between perfect squares and prime numbers have remained one of the most aesthetically appealing areas of mathematics.

In the literature on number-theoretic graphs, many structures have been proposed in which primes serve as vertices under various algebraic relations; prime-difference graphs, factorization graphs, and Cayley graphs based on modular arithmetic are examples of such approaches (Ribenboim, 2004; West, 2001). However, the majority of these studies focus on the multiplicative or modular properties of primes rather than their additive properties. To the authors' knowledge, no study in the literature has systematically examined a connectivity relation based on a perfect-square-sum condition; this constitutes the original contribution of the present work.

In this study, unlike traditional approaches in the literature, the algebraic properties of numbers are transformed into the geometric bonds (edges) of points in a space. This approach allows the algebraic boundaries of numbers to be analysed visually and topologically.

The remainder of the paper is organized as follows. Section 2 formally defines the Quadratic Prime Graph and supports it with examples. Section 3 proves the main theorem (the 3–7 Common Neighbour Impossibility). Section 4 extends this result to the general case $|p - q| = 4$ and applies it to cousin prime pairs. Section 5 discusses the findings, addresses the limitations of the study, and proposes directions for future research.

2. Basic Concepts and Definitions

The mathematical space underlying this study is constructed as follows.

Definition 2.1 (Quadratic Prime Graph, G_Q). Let P denote the set of all positive primes, and let us define a simple undirected graph $G_Q = (V, E)$.

Vertices (V): Each vertex in the graph represents a unique prime number $v \in P$.

Edges (E): An edge is drawn between two primes p_i and p_j ($i \neq j$) if and only if their sum forms a perfect square. That is:

$$e = \{p_i, p_j\} \in E \Leftrightarrow p_i + p_j = k^2, k \in \mathbf{Z}^+$$

Remark 2.1. The relation given in Definition 2.1 is symmetric, since $p_i + p_j = p_j + p_i$; hence G_Q is an undirected graph. Since the condition $i \neq j$ is required in the definition, the case $p = 2$, where $2 + 2 = 4 = 2^2$, does not create a self-loop; this special case is excluded from the scope of the definition, and G_Q retains the property of being a simple graph.

Example 2.1. Following the proposed definition, some connections are as follows:

$$\{3, 13\} \in E, \text{ since } 3 + 13 = 16 = 4^2$$

$$\{13, 23\} \in E, \text{ since } 13 + 23 = 36 = 6^2$$

$$\{23, 2\} \in E, \text{ since } 23 + 2 = 25 = 5^2$$

$$\{2, 7\} \in E, \text{ since } 2 + 7 = 9 = 3^2$$

Thus, a path 3 – 13 – 23 – 2 – 7 exists in the graph G_Q.

3. Main Theorem and Proof

One of the first topological questions that arises is whether closed triangles (3-cycles) can be formed on this graph. For instance, is there a vertex connected to both 3 and 7 simultaneously? The following theorem establishes that this is algebraically impossible.

Theorem 3.1 (3–7 Common Neighbour Impossibility). In the graph G_Q, there exists no vertex $X \in P$ (or $X \in \mathbf{Z}^+$) that shares an edge with both 3 and 7 simultaneously.

Proof. We use proof by contradiction. Suppose there exists a number $X \in \mathbf{Z}^+$ connected to both 3 and 7. By Definition 2.1, the following two conditions must hold simultaneously:

$$(1) \;\; X + 3 = A^2, \;\; A \in \mathbf{Z}^+$$

$$(2) \;\; X + 7 = B^2, \;\; B \in \mathbf{Z}^+$$

Subtracting equation (1) from equation (2):

$$(X + 7) - (X + 3) = B^2 - A^2 \Longrightarrow 4 = B^2 - A^2$$

Applying the difference of squares identity:

$$(B - A)(B + A) = 4$$

Since A and B are positive integers, $(B - A)$ and $(B + A)$ are also integers, and $(B + A) > (B - A)$ holds. Furthermore, since $(B + A) + (B - A) = 2B$ is even, the two factors must share the same parity (either both odd or both even); this observation shows that the factor pair (1, 4), which has mismatched parity, can be ruled out directly. For completeness, the proof examines both cases:

Case 1: Factors 1 and 4. Adding $B - A = 1$ and $B + A = 4$ gives $2B = 5 \Longrightarrow B = 2.5$. Since B must be an integer, this is a contradiction (as the parity mismatch already indicates).

Case 2: Factors 2 and 2. Solving $B - A = 2$ and $B + A = 2$ gives $2B = 4 \Longrightarrow B = 2$ and $A = 0$. Substituting $A = 0$ into equation (1):

$$X + 3 = 0^2 \Longrightarrow X = -3$$

is obtained. By the definition of the graph, the vertex X must be a positive prime (or positive natural number). A negative vertex value contradicts the initial assumption. Therefore, the equation has no solution for $X \in \mathbf{Z}^+$. ■

4. Generalization of the Theorem (Generalized 4-Difference Theorem)

Examining the proof method of Theorem 3.1 reveals that the element leading to the impossibility is not the specific numbers 3 and 7, but the fact that their difference is 4. This observation allows the theorem to be cast in a much stronger form.

Theorem 4.1 (Generalized 4-Difference Impossibility Theorem). Let p and q be any two vertices in the Quadratic Prime Graph (G_Q). If $|p - q| = 4$, then these two vertices can have no common neighbour (i.e., no 3-cycle, C_3, can be formed through these two vertices).

Proof. Without loss of generality, assume $q > p$ and $q - p = 4$. Let X be a common neighbour:

$$X + p = A^2, \;\; X + q = B^2$$

Subtracting:

$$q - p = B^2 - A^2 \Longrightarrow 4 = B^2 - A^2$$

By the Case 2 analysis in Theorem 3.1, the only integer solution satisfying this equality is $A = 0$ and $B = 2$. If $A = 0$, then $X + p = 0 \Longrightarrow X = -p$. Since p is a positive prime, $-p$ is always negative. Therefore, no positive solution for X can exist. ■

Corollary 4.1. According to this generalization, not only 3 and 7, but none of the "Cousin Prime" pairs differing by 4, such as (19, 23), (37, 41), (67, 71), (79, 83), can form a triangle with any other number in this quadratic topology.

5. Conclusion, Discussion, and Limitations

In this paper, a new network structure has been constructed by synthesizing number theory and topology. Using basic Diophantine equations, impassability boundaries between specific vertices in the Quadratic Prime Graph (particularly primes differing by 4) have been proven. The result shows that cousin prime pairs always exhibit a triangle-free local structure within G_Q, providing an example of how triangle-free graph families can be generated by purely algebraic means.

The scope of this study is limited to an algebraic proof carried out over a finite set of examples; statistical properties of the G_Q graph, such as its asymptotic density, average degree distribution, or general connectivity behaviour, are left outside the scope of this work. Furthermore, the result has only been proven for the special case $|p - q| = 4$, and the general case $|p - q| = 2n$ remains an open problem.

Future research may address the following questions:

1. Which prime number has the highest degree in the graph G_Q?

2. Does the Quadratic Prime Graph form a single infinite connected component, or does it consist of independent isolated islands?

3. Can Theorem 4.1 be generalized to even differences other than $|p - q| = 4$ (e.g., $2n$, $n \in \mathbf{Z}^+$), and for which difference values is triangle formation possible?

Ethics Statement and Additional Information

Ethics Committee Approval: This study is a theoretical/analytical mathematics investigation that does not involve any experimentation on human or animal subjects; therefore, ethics committee approval is not required.

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References

Bacak, G., & Beşeri, T. (2002). An overview of graph theory. *Matematik Dünyası*, 11(4), 15–20.

Diestel, R. (2017). *Graph theory* (5th ed.). Springer-Verlag.

Hardy, G. H., & Wright, E. M. (2008). *An introduction to the theory of numbers* (6th ed.). Oxford University Press.

Nuriyev, U. G., & Sadıgova, H. (2002). On twin prime numbers. *Matematik Dünyası*, 11(4), 8–14.

Ribenboim, P. (2004). *The little book of bigger primes* (2nd ed.). Springer.

West, D. B. (2001). *Introduction to graph theory* (2nd ed.). Prentice Hall.