Quantum Identity Constant

Ξ

Definition, Convergence Analysis, and Irrationality Assessment of an Irrational Constant

Keywords: *irrational constants, infinite series, Liouville's theorem, Bose–Einstein statistics, quantum statistical mechanics*

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Abstract

This study defines a new mathematical constant based on the hyper-exponential growth (n^n) that characterizes the macroscopic world and the factorial growth ($n!$) that characterizes the quantum world. Named the Quantum Identity Constant (Ξ), this constant is defined by the following infinite series, in which these two growth rates appear together in the denominator:

$$\boldsymbol{\Xi = \Sigma\ (n=1 \to \infty)\ 1 / (n^n + n!) \approx 0.700872\ldots}$$

The convergence of the series is examined. A Liouville-type proof strategy is attempted to show that Ξ cannot be rational; this English revision identifies a growth-rate error that invalidates the argument as originally stated (see Section 4.3) and reclassifies the result as an open conjecture supported by the same numerical and structural evidence used to motivate the original claim. Possible scientific applications are also discussed.

1. Introduction and Motivation

The emergence of new constants in mathematics most often arises from two different growth regimes, or two different world-views, meeting within a single equation. The constant *e* arose from the continuous compounding of interest; π arose from the cyclical structure of a circle's geometry. With a similar motivation, this study carries a fundamental asymmetry between the macroscopic and quantum worlds into a mathematical framework.

In the macroscopic world, objects are individual; no matter how similar two physical objects are, they remain distinguishable from one another. In classical statistics, the number of distinct arrangements of n distinguishable objects grows proportionally to n^n (hyper-exponential speed). In quantum mechanics, by contrast, identical particles — two electrons, for instance — are fundamentally indistinguishable from one another. The number of ways such particles can be permuted is measured by $n!$ (factorial speed).

Considering these two growth rates together in the following infinite series raises a natural question: what is the sum of this series, and is this value irrational?

2. Mathematical Definition

The Quantum Identity Constant Ξ (capital xi) is defined by the following infinite series:

Definition 1.

$$\boldsymbol{\Xi = \Sigma(n=1 \to \infty)\ 1 / (n^n + n!)}$$

Here $\mathbf{n^n}$ denotes the hyper-exponential term, and **n!** denotes the factorial term.

Writing the general term of the series as $a\_n = 1/(n^n + n!)$, the series expands as:

$$\Xi = 1/(1+1) + 1/(4+2) + 1/(27+6) + 1/(256+24) + 1/(3125+120) + \cdots$$

$$\Xi = 1/2 + 1/6 + 1/33 + 1/280 + 1/3245 + \cdots$$

3. Convergence Analysis and Approximate Value

To determine whether the series converges, we apply the comparison and ratio tests.

3.1. Comparison Test

Since $n^n \geq 1$ for $n \geq 1$, the denominator satisfies $n^n + n! \geq n!$. Therefore:

$$0 < a\_n = 1/(n^n + n!) \leq 1/n!$$

Since the series $\Sigma(1/n!) = e - 1 \approx 1.71828\ldots$ converges, Ξ also converges by the comparison test. Hence $\Xi \in (0, e-1)$.

3.2. Numerical Convergence Table

The table below shows the first five partial sums:

n	n^n	n!	$a_n = 1/(n^n + n!)$	S_n (cumulative)
1	$1^1 = 1$	1! = 1	1/2 = 0.5	0.500000
2	$2^2 = 4$	2! = 2	1/6 ≈ 0.16667	0.666667
3	$3^3 = 27$	3! = 6	1/33 ≈ 0.03030	0.696970
4	$4^4 = 256$	4! = 24	1/280 ≈ 0.00357	0.700540
5	$5^5 = 3125$	5! = 120	1/3245 ≈ 0.00031	0.700851

As can be seen, the general terms approach zero extremely quickly, and the contribution added from the fifth step onward drops below the order of 10^{-3}. However, achieving full convergence to six correct decimal digits requires several further terms beyond n = 5.

Remark (Corrected Numerical Value). Recomputing the series with arbitrary-precision arithmetic (mpmath, 60 decimal digits) through n = 50 shows the sum is fully stable from around n = 30 onward at:

$$\Xi \approx 0.7008716687086434707\ldots$$

This refines the original estimate of 0.700851… (which corresponds only to the fifth partial sum, S_5, and had not yet converged to that many digits) to the fully converged value 0.700872 (to six decimals). The n = 6 and n = 7 terms contribute an additional +0.0000211 and +0.0000012 respectively, accounting for the difference. This correction does not affect the convergence argument in Section 3.1, which remains valid regardless of the precise decimal value.

4. Assessment of Irrationality

This section attempts to show that Ξ is irrational via an application of Liouville's theorem. As detailed in Section 4.3, the specific growth-rate comparison used in the original argument does not hold, so the result is presented here as an open conjecture rather than a completed proof.

4.1. Liouville's Theorem (1844)

Theorem (Liouville, 1844).
If α is an algebraic real number of degree d ≥ 1, then there exists a constant C > 0 such that for every rational number p/q (q > 0),

$$|\alpha - p/q| \geq C / q^d$$

the inequality holds. In other words, algebraic numbers cannot be approximated 'too quickly' by rational numbers. If a number's rational approximations violate this protective bound, the number cannot be algebraic (and hence cannot be rational).

4.2. Application of the Proof

Let the N-th partial sum be $S\_N = \Sigma(k=1 \to N)\ 1/(k^k + k!)$. Let us bound the tail of the series:

$$\Xi - S\_N = \Sigma(k=N+1 \to \infty)\ 1/(k^k + k!) < \Sigma(k=N+1 \to \infty)\ 1/k^k$$

Owing to the dominant behavior of the hyper-exponential term, this tail is bounded by roughly twice its leading term:

$$\Xi - S\_N < 2 / (N+1)^{(N+1)}$$

On the other hand, S_N can be brought to a common denominator $Q\_N = \prod(k=1 \to N)\ (k^k + k!)$, a large but finite integer, so that $S\_N = P\_N/Q\_N$.

Now suppose Ξ = p/q is rational. Then, since $\Xi \neq S\_N$ for any finite N (the remaining terms are strictly positive), the standard rational-approximation lower bound gives:

$$|\Xi - P\_N/Q\_N| \geq 1 / (q \cdot Q\_N)$$

Combining this with the upper bound from the tail estimate:

$$1 / (q \cdot Q\_N) < 2 / (N+1)^{(N+1)}$$

that is, $(N+1)^{(N+1)} < 2q \cdot Q\_N$. The original argument claimed that Q_N grows at most on the order of $(N!)^N$, while $(N+1)^{(N+1)}$ grows *faster* — and concluded that this inequality must eventually fail, yielding a contradiction that would prove Ξ irrational.

4.3. A Growth-Rate Error in the Original Argument

Careful asymptotic analysis shows the opposite is true. Since $\ln(Q\_N) = \Sigma(k=1 \to N) \ln(k^k + k!) \approx \Sigma(k=1 \to N)\, k \ln k \approx (N^2/2) \ln N$ for large N, while $\ln((N+1)^{(N+1)}) = (N+1)\ln(N+1) \approx N \ln N$, the denominator Q_N in fact grows on the order of $N^2 \ln N$ in its logarithm — dramatically *faster* than $(N+1)^{(N+1)}$, whose logarithm grows only on the order of $N \ln N$. This has been confirmed numerically: already at $N = 5$, $\ln(Q\_N) \approx 19.7$ versus $\ln(2(N+1)^{(N+1)}) \approx 11.4$, and the gap widens rapidly with N (at $N = 20$, $\ln(Q\_N) \approx 531$ versus $\ln(2(N+1)^{(N+1)}) \approx 65$).

This means the inequality $(N+1)^{(N+1)} < 2q \cdot Q\_N$ holds *easily* — not marginally, but by an ever-widening margin — for every sufficiently large N, regardless of the value of q. No contradiction is ever reached by this route: the natural common denominator Q_N of the partial sums grows so much faster than the tail bound requires that the Liouville-style comparison is too loose to detect any obstruction to rationality. This is a genuine error in the original argument's asymptotic comparison, not merely an imprecise estimate, and it invalidates the proof as stated.

4.4. Conjecture and Discussion

Conjecture 4.1 (Irrationality of Ξ). Ξ is conjectured to be irrational, and plausibly transcendental, though the argument in Sections 4.1–4.2 does not establish this.

The underlying difficulty is structural: the individual terms $1/(n^n+n!)$ already decay extremely fast (a form of 'built-in' rapid convergence), but this very speed means that the natural common denominator of the first N terms — a *product* of N rapidly growing integers — grows even faster still, on a quadratic-in-N exponential scale, outpacing any Liouville-type tail bound derived from a single dominant term. A successful proof would likely require either (i) a smarter choice of auxiliary denominator (e.g., a suitably constructed integer smaller than the full product Q_N while still clearing all relevant denominators), or (ii) an entirely different Diophantine approximation technique tailored to the specific structure of $n^n + n!$. Both directions remain open (see Section 6).

5. Possible Scientific Applications

As in the original study, the interpretations below are offered as speculative conceptual bridges rather than established physical results.

Because Ξ structurally unites two growth regimes, it may offer a theoretical framework in the following areas:

- **Bose–Einstein Condensation:** As absolute zero is approached, atoms lose their individuality and collapse into a shared quantum state. The disorder–identity competition in this transition could be formalized through Ξ via its n^n and $n!$ denominator terms.

- **Data Compression Theory:** The theoretical upper bound on the efficiency of merging identical blocks in compression algorithms, for infinite and fully random data sets, might be related to Ξ.

- **Quantum Statistical Mechanics:** Ξ could serve as an integrated measure of the indistinguishability corrections appearing in Bose–Einstein and Fermi–Dirac distributions.

6. Conclusion

This study has defined a new constant Ξ that unites the hyper-exponential (n^n) and factorial ($n!$) growth rates within a common denominator, and has presented a numerical convergence analysis together with a comparison-test proof of convergence. The fully converged value has been determined via high-precision computation to be $\Xi \approx 0.700872$ (refining the original five-term estimate of 0.700851).

The Liouville-based irrationality argument presented in the original study, however, contains a genuine asymptotic error: the natural common denominator of the partial sums grows far faster than the tail bound it is compared against, so the claimed contradiction does not arise. The irrationality — and *a fortiori* the transcendence — of Ξ is accordingly presented here as an open conjecture rather than a proven theorem. Establishing this rigorously, whether via a refined Liouville-type argument with a better-chosen auxiliary denominator, or via the Gelfond–Schneider theorem or Baker's linear-independence criteria, remains a priority open question for future work.

References

Cover, T. M., & Thomas, J. A. (2006). *Elements of Information Theory* (2nd ed.). Wiley-Interscience.

Hardy, G. H., & Wright, E. M. (2008). *An Introduction to the Theory of Numbers* (6th ed.). Oxford University Press.

Liouville, J. (1844). Sur des classes très étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationelles algébriques. *Journal de Mathématiques Pures et Appliquées*, 16, 133–142.

Niven, I. (1956). *Irrational Numbers*. Mathematical Association of America.

Pathria, R. K., & Beale, P. D. (2011). *Statistical Mechanics* (3rd ed.). Elsevier.