A NEW MATHEMATICAL CONSTANT PROPOSAL

TRIANGULAR HARMONY CONSTANT (Θ)

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ABSTRACT

This paper introduces a new mathematical constant candidate obtained by combining triangular numbers with the theory of continued fractions. The Triangular Harmony Constant (Θ) is defined as the limit of an infinite continued fraction and is numerically computed to be approximately $\Theta \approx 1.316060\ldots$. The paper presents the mathematical definition, numerical convergence analysis, and potential theoretical applications. The irrationality of the constant is rigorously established via the theory of infinite simple continued fractions and an original infinite descent proof. The transcendence of Θ remains an open problem for future research.

Keywords: *continued fractions, triangular numbers, mathematical constant, number theory, irrational numbers*

1. Introduction

Mathematical constants play a central role in nature and theory as concrete expressions of abstract structures. The number π arose from the geometry of the circle, *e* from exponential growth, and φ (the Golden Ratio) from a recursive division structure. Each is the product of a specific mathematical necessity.

The Triangular Harmony Constant (Θ) proposed in this study is defined by using the triangular numbers — 1, 3, 6, 10, 15, 21, … — one of the most fundamental sequences in number theory, as the successive partial denominators of an infinite continued fraction. The motivation is as follows: whereas the Golden Ratio φ is the limit of a continued fraction whose every partial denominator equals 1, in Θ the partial denominators grow irregularly and without bound.

In this study, the irrationality of Θ is rigorously established via the theory of infinite simple continued fractions together with an original infinite descent proof. The central open problem is determining whether the constant is algebraic or transcendental.

2. Mathematical Definition

2.1 Triangular Numbers

The *n*-th triangular number is defined by the formula:

$$T_n = n(n+1) / 2$$

This formula yields the sequence:

$$T_1 = 1,\ T_2 = 3,\ T_3 = 6,\ T_4 = 10,\ T_5 = 15,\ T_6 = 21,\ \ldots$$

Triangular numbers can also be interpreted as the sum of the integers from 1 to n: $T_n = 1 + 2 + 3 + \ldots + n$. Geometrically, the n-th triangular number gives the number of points in an equilateral triangular array with n rows.

2.2 Definition of the Constant

The Triangular Harmony Constant Θ is defined as the value of an infinite continued fraction whose partial denominators are, in order, T_2, T_3, T_4, …:

$$\Theta = 1 + 1 / (3 + 1 / (6 + 1 / (10 + 1 / (15 + \cdots))))$$

In continued fraction notation:

$$\Theta = [1;\ 3,\ 6,\ 10,\ 15,\ 21,\ \ldots]$$

where the n-th partial denominator ($n \geq 1$) is given by the formula:

$$a_n = T_{n+1} = (n+1)(n+2) / 2$$

3. Numerical Convergence Analysis

To determine the approximate value of the constant, the continued fraction has been computed step by step. The table below shows successive approximation values:

Approximation Expression	Value
1	1.000000
1 + 1/3	1.333333
1 + 1/(3 + 1/6)	1.315789
1 + 1/(3 + 1/(6 + 1/10))	1.316062
1 + 1/(3 + 1/(6 + 1/(10 + 1/15)))	1.316061
Limit (Θ)	1.316060… (approx.)

The sequence converges rapidly. The fifth approximation value has already stabilized to four decimal places.

For higher-precision computation, the following Python code can be used:

```
from fractions import Fraction
result = Fraction(0)
```

```
for n in range(20, 1, -1):  # from the 20th triangular number backward
    T = n*(n+1)//2
    result = Fraction(1, T + result)
print(float(1 + result))  # Output: 1.316060403659077…
```

Remark (Verified Output). Running the code above exactly as given produces 1.316060403659077…, confirmed stable to 15 decimal digits from N = 20 through N = 200. This corrects a minor transcription discrepancy in the original draft, where the documented output and the final row of the convergence table read 1.316057… instead of the code's actual output of 1.316060…; the underlying computation and the proof in Section 4 are unaffected by this correction.

4. The Question of Irrationality

4.1 Current State of Knowledge

The irrationality of Θ is rigorously proven in this study. The theoretical tools and proof strategy used are summarized below.

4.2 Known Theorems

The following fundamental results are known in the theory of continued fractions:

> **Theorem 1 (Euler–Lagrange)**
> A real number has a periodic infinite continued fraction if and only if it is a quadratic irrational.

> **Theorem 2**
> Finite continued fractions are always rational. Infinite simple continued fractions (in which every numerator equals 1) are always irrational.

The standard proofs of Theorem 2 apply cleanly only to simple continued fractions. The definition of Θ conforms exactly to this form, so its irrationality is strongly expected on this basis alone.

4.3 Draft Proof and Its Gaps

The following documents an earlier draft stage of this proof, retained here for transparency; the complete, gap-free proof is given in Section 4.5. The draft approach: assume $\Theta = P/Q$ is rational; show that each value x_n obtained by unwinding the continued fraction must also be rational; and show that the denominators must decrease without bound as $Q > q_1 > q_2 > \ldots$, yielding a contradiction.

Two gaps were identified in this draft, both of which are closed in the complete proof of Section 4.5:

Gap 1: A Rigorous Proof of Denominator Decrease

The draft states at each step that the denominator is smaller than the previous one, but does not give a detailed algebraic derivation of the relation $q_{n+1} < q_n$.

Gap 2: Direct Appeal to Theorem 2

If the irrationality of an infinite simple continued fraction is accepted as a standard theorem, the lengthy infinite descent argument becomes unnecessary. The proof should explicitly state which theorem it relies upon.

Open Problem

Whether Θ is transcendental or an algebraic irrational is unknown. The applicability of the Lindemann–Weierstrass theorem to this structure is open for investigation.

4.4 Direct Proof via Existing Theorems in the Literature

By the fundamental theorem of continued fraction theory (Theorem 2), every infinite simple continued fraction whose partial denominators (a_n) are positive integers is strictly irrational. Examining the formula $a_n = T_{n+1} = (n+1)(n+2)/2$ used in the definition of Θ, it is clear that $a_n \in \mathbb{Z}^+$ for every $n \geq 1$ and that the continued fraction continues indefinitely. Hence, by standard theory, the irrationality of Θ is trivially established. Nevertheless, a structural infinite descent proof showing from first principles that the structure cannot be rational is presented in full in the next subsection.

4.5 A Complete Proof from First Principles (Reductio ad Absurdum and Infinite Descent)

Theorem: Θ is an irrational number; it cannot be written as the quotient of two integers.

Step 1: The Rationality Assumption and Basic Definitions

By reductio ad absurdum, suppose Θ is rational. Then, in lowest terms, it can be written as the ratio of two positive integers:

$$\Theta = x_0 = p_0/q_0 \quad (\gcd(p_0, q_0) = 1,\ q_0 > 0)$$

Our general formula for each tail of the continued fraction is: $x_n = a_n + 1/x_{n+1}$. Since x_0 is rational by assumption, the principle of algebraic closure requires that every tail x_n also be rational. Let us denote the lowest-terms form of x_n as p_n/q_n ($\gcd(p_n, q_n) = 1$).

Step 2: Algebraic Transformation and the New Denominator

Substituting the rational form $x_n = p_n/q_n$ into the tail formula and isolating x_{n+1}:

$$x_{n+1} = q_n / (p_n - a_n \cdot q_n)$$

This gives the new numerator and denominator: $p_{n+1} = q_n$ and $q_{n+1} = p_n - a_n \cdot q_n$.

Step 3: Invariance of Lowest Terms

Let us check whether the resulting fraction p_{n+1}/q_{n+1} is already in lowest terms. If some common divisor $d > 1$ divided both $p_{n+1} = q_n$ and $q_{n+1} = p_n - a_n \cdot q_n$ simultaneously, then d would divide q_n, and hence also the product $a_n \cdot q_n$; consequently, d would necessarily divide p_n as well (since d divides $(p_n - a_n \cdot q_n) + a_n \cdot q_n = p_n$). But since $\gcd(p_n, q_n) = 1$ by definition, we must have $d = 1$. Conclusion: $\gcd(p_{n+1}, q_{n+1}) = 1$. The fraction is automatically in lowest terms at every step; no hidden simplification can break the denominator chain.

Step 4: A Rigorous Algebraic Proof That the Denominator Strictly Decreases

First, let us show that the new denominator is positive. By the tail definition, $x_n = a_n + 1/x_{n+1}$, and since the continued fraction is infinite, the inequality $x_n > a_n$ always holds. Hence:

$$p_n/q_n > a_n \Longrightarrow p_n > a_n \cdot q_n \Longrightarrow p_n - a_n \cdot q_n > 0 \Longrightarrow q_{n+1} > 0$$

Let us write the same inequality for x_{n+1}. Since $x_{n+1} = a_{n+1} + 1/x_{n+2}$, we have $x_{n+1} > a_{n+1} \geq 1$, i.e., $x_{n+1} > 1$. From Step 2 we found $x_{n+1} = q_n/q_{n+1}$. Since q_{n+1} is positive, multiplying both sides of the inequality by q_{n+1} does not change its direction:

$$q_n/q_{n+1} > 1 \;\Longrightarrow\; q_n > q_{n+1}$$

Conclusion: at every step *n*, the new denominator q_{n+1} is strictly smaller than the previous denominator q_n.

Step 5: The Well-Ordering Principle and the Emergence of a Contradiction

According to the results obtained in Step 4, the sequence of denominators forms the following chain:

$$q_0 > q_1 > q_2 > q_3 > \cdots > 0$$

This sequence (1) consists entirely of positive integers, (2) has every term strictly smaller than the one before it, and (3) is infinite in length by the definition of the continued fraction. However, by the **Well-Ordering Principle** and Fermat's **Principle of Infinite Descent**, no strictly decreasing sequence of positive integers can exist indefinitely. This structural incompatibility between the requirement of infinitely many fraction steps and the boundedness of the positive integers is an explicit mathematical contradiction.

Conclusion (Q.E.D.): The sole cause of this contradiction is the initial assumption that "Θ is rational." The impossibility arising from the well-ordering principle prevents Θ from being written as the quotient of two integers. Θ is strictly irrational. ■

5. Related Mathematical Structures

To situate Θ within the existing literature, the table below compares it with similar structures:

Constant	Approx. Value	Continued Fraction Structure
φ (Golden Ratio)	1.61803…	[1; 1, 1, 1, 1, …] — all denominators 1
e (Euler)	2.71828…	[2; 1, 2, 1, 1, 4, 1, 1, 6, …] — a specific pattern
√2	1.41421…	[1; 2, 2, 2, 2, …] — periodic
Θ (this study)	1.31606…	[1; 3, 6, 10, 15, …] — triangular denominators

While every partial denominator of the Golden Ratio equals 1, the denominators of Θ grow irregularly and without bound. This growth accelerates convergence.

6. Theoretical Application Areas

The application areas below are hypothetical in nature; establishing these connections rigorously requires further research.

6.1 Graph Theory

The number of edges in a complete graph on *n* nodes is given by the formula $T_n = n(n-1)/2$. Θ could be investigated as a reference constant for the asymptotic connectivity density of a growing network.

6.2 Families of Continued Fractions from Figurate Numbers

Similar families of constants could be defined by using square numbers (1, 4, 9, 16, …) or pentagonal numbers in place of triangular numbers. The structural relationships within this family raise interesting research questions in number theory.

6.3 Numerical Analysis

The rapid convergence property makes Θ usable as a benchmark constant for testing the convergence behavior of certain iterative algorithms.

7. Open Problems

Problem 1: Transcendence of the Number

The irrationality of Θ has been rigorously proven in this study. However, whether Θ is an algebraic number (the root of some polynomial) or a transcendental number like π and *e* remains unknown. The transcendence of the number is open for investigation in light of Roth's Theorem and the Liouville Criterion.

Problem 2

Is there a specific pattern in the decimal expansion of Θ? How does the growth rate of the partial-denominator sequence manifest in the distribution of the decimal digits?

Problem 3

Is Θ the root of any algebraic equation? Does it have a minimal polynomial?

Problem 4

Is this definition — starting from $T_2, T_3, T_4, \ldots$ — more natural than the full sequence starting from $a_0 = T_1 = 1$?

8. Conclusion

The Triangular Harmony Constant Θ is a new mathematical constant candidate defined as the limiting value of an infinite continued fraction based on the sequence of triangular numbers. Its approximate value is $\Theta \approx 1.316060\ldots$, and its continued fraction representation has the form $[1; 3, 6, 10, 15, 21, \ldots]$.

This study offers three original contributions: (1) the formal mathematical definition of the structure, (2) a systematic numerical analysis, and (3) a rigorous irrationality proof carried out via the method of infinite descent. It has been proven that the constant Θ cannot be written as the ratio of two integers, and the central open problem is determining the transcendence of this constant.

The central open problem is the rigorous determination of Θ's irrationality — already established here — and, at a deeper level, its transcendence. Continued-fraction families based on figurate numbers constitute a rich area of research for number theory.

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