The Additive Distribution of Twin Primes on Eulerian Graphs: The S = 24|E| Theorem

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Abstract

This paper proposes an algebraic relation, not previously examined in the literature, between twin primes—a long-standing subject of number theory—and Eulerian graphs from topological graph theory. A new structure, the "Twin-Prime Graph" (G_{TP}), is defined, in which the degree of each vertex is linked to the generator value n of the twin-prime pair it represents (of the form $6n \pm 1$). It is shown algebraically that, whenever this graph satisfies the Eulerian condition (all vertex degrees even), the total sum of all twin primes represented in the graph equals exactly 24 times the number of edges ($S = 24|E|$). The result follows directly from the Handshaking Lemma combined with the well-known $6n \pm 1$ representation of twin primes, and shows that the sum of large sets of twin primes can be obtained solely from the edge count, without summing the primes individually.

Keywords

Twin Primes; Graph Theory; Eulerian Graph; Handshaking Lemma; Number Theory.

Öz

Bu makalede, sayılar teorisinin köklü araştırma alanlarından biri olan ikiz asal sayılar ile topolojik çizge kuramındaki Euler çizgeleri arasında, literatürde doğrudan incelenmemiş cebirsel bir bağıntı önerilmektedir. Çalışmada "İkiz-Asal Çizgesi" (G_{TP}) adı verilen yeni bir yapı tanımlanmış; köşelerin dereceleri, temsil ettikleri ikiz asal çiftinin üreteç değeri n ile ilişkilendirilmiştir. Bu çizge yapısının Euler koşullarını sağladığı özel durumda, çizgedeki tüm ikiz asal sayıların genel toplamının, çizgenin ayrıt sayısının tam olarak 24 katına eşit olduğu ($S = 24|E|$) cebirsel olarak ispatlanmıştır.

Anahtar Kelimeler

İkiz Asal Sayılar; Çizge Kuramı; Euler Çizgesi; Tokalaşma Teoremi; Sayılar Teorisi.

1. Introduction

In mathematical research, Number Theory and Graph Theory are generally treated as distinct fields of study. Graph theory is primarily concerned with topological optimization among points and connections, while number theory focuses on the distribution of integers (particularly primes) and the rules of prime factorization. Studies bridging these two fields—for example, Cayley graphs with prime vertices or network structures based on modular arithmetic—have received growing attention in the literature (West, 2001; Diestel, 2017), yet no example has been found that jointly addresses the additive properties of twin primes and the structural conditions of Eulerian graphs.

The purpose of this study is to construct a synthesis theorem between the $6n \pm 1$ arrangement of twin primes formulated by Nuriyev and Sadıgova (2002) and the Eulerian graphs underlying the Königsberg Bridge Problem, as examined by Bacak and Beşeri (2002). In this context, a new graph topology is defined in which vertices represent twin-prime pairs and degrees are determined by the arithmetic properties of these primes; a striking additive regularity is demonstrated in the special case where this structure satisfies the Eulerian condition.

The remainder of the paper is organized as follows. Section 2 recalls twin primes, graph/degree concepts, and the Handshaking and Eulerian theorems. Section 3 defines the Twin-Prime Graph. Section 4 proves the main theorem ($S = 24|E|$). Section 5 discusses the findings and addresses the study's limitations and directions for future research.

2. Basic Concepts and Preliminaries

Definition 2.1 (Twin Primes). Pairs of primes differing by 2 are called twin primes. With the exception of the pair (3, 5), all twin prime pairs for $p > 3$ are of the form $p = 6n - 1$ and $q = 6n + 1$, where $n \in \mathbf{Z}^+$ (Nuriyev & Sadıgova, 2002). Hence, the sum of every twin-prime pair is:

$$p + q = 12n$$

Definition 2.2 (Graph and Degree). Let $G = (V, E)$ be a connected graph, where V denotes the vertex set and E the edge set. The number of edges incident to a vertex v_i is called the degree of that vertex, denoted $\deg(v_i)$.

Theorem 2.1 (Handshaking Lemma). In any graph $G(V, E)$, the sum of the degrees of all vertices equals twice the number of edges (Bacak & Beşeri, 2002):

$$\Sigma_{v \in V} \deg(v) = 2|E|$$

Theorem 2.2 (Eulerian Graph — Euler, 1736). A necessary and sufficient condition for a connected graph $G(V, E)$ to be Eulerian (i.e., admitting a closed tour traversing every edge exactly once) is that every vertex in the graph has even degree:

$$\forall v_i \in V, \deg(v_i) \equiv 0 \pmod 2$$

3. A New Structure: The Twin-Prime Graph

In light of the basic definitions above, a new graph structure is constructed.

Definition 3.1 (Twin-Prime Graph). Let $G_{TP} = (V, E)$ be a simple, undirected, connected graph in which each vertex v_i represents a unique twin-prime pair (p_i, q_i) greater than 3. The degree of each vertex v_i in G_{TP} is defined as one-sixth of the arithmetic mean of the twin-prime pair it represents:

$$deg(v_i) = (p_i + q_i) / 2 \div 6 = (p_i + q_i) / 12$$

Remark 3.1. By Definition 2.1, $p_i + q_i = 12n_i$, so $\deg(v_i) = n_i \in \mathbf{Z}^+$ is always a positive integer; hence the definition is consistent with the fundamental requirement of graph theory that degrees be natural numbers. This shows that the Twin-Prime Graph is a well-defined structure.

4. Main Theorem and Proof

Theorem 4.1 (Total Value in Eulerian Twin-Prime Graphs). Let $G_{TP} = (V, E)$ be a Twin-Prime Graph constructed according to Definition 3.1. If G_{TP} satisfies the Eulerian graph conditions, then the total sum of all twin primes in this graph (S) is exactly equal to 24 times the total number of edges ($|E|$):

$$S = \Sigma_{i=1}^{|V|} (p_i + q_i) = 24|E|$$

Proof.

1. By Definition 2.1, the sum of the twin-prime pair at each vertex v_i is $p_i + q_i = 12n_i$.

2. By the degree rule in Definition 3.1, the vertex degree is:

$$deg(v_i) = 12n_i / 12 = n_i$$

Hence, the degree of the vertex equals the generator parameter n_i of the primes.

3. The theorem assumes that G_{TP} is Eulerian. By Theorem 2.2, all vertex degrees in an Eulerian graph must be even. This forces n_i to be even, so $n_i = 2k_i$ for some $k_i \in \mathbf{Z}^+$.

4. Let us write the total prime sum (S) in the graph:

$$S = \Sigma (p_i + q_i) = \Sigma\, 12n_i$$

Substituting $n_i = 2k_i$:

$$S = \Sigma\, 12(2k_i) = 24\, \Sigma\, k_i \quad (Equation\ I)$$

5. Now applying the Handshaking Lemma (Theorem 2.1): since $\Sigma \deg(v_i) = 2|E|$ and $\deg(v_i) = n_i = 2k_i$:

$$\Sigma\, 2k_i = 2|E|$$

Dividing both sides by 2:

$$\Sigma\, k_i = |E| \quad (Equation\ II)$$

6. Substituting the result of Equation II into Equation I:

$$S = 24|E| \ \blacksquare$$

5. Conclusion, Discussion, and Limitations

In this study, a synthesis theorem has been presented between the disciplines of Number Theory and Graph Theory. The resulting $S = 24|E|$ theorem is notable in that it shows that the sum of twin primes can be obtained solely from graph topology (the edge count), without computing the numbers themselves: to find the sum of the prime values contained in a closed, traversable (Eulerian) network of twin primes with hundreds of thousands of digits, there is no need to compute or sum the primes individually; it suffices to count the edges topologically and multiply the result by 24.

The validity of this result depends on the special case where the graph G_{TP} satisfies the Eulerian condition; the sum formula cannot be directly applied to general (non-Eulerian) Twin-Prime Graphs, and in that case S must be expressed differently, depending on the parity distribution of vertex degrees. Furthermore, the study assumes a graph structure defined over finite, previously known sets of twin primes and makes no claim regarding the truth or falsity of the Twin Prime Conjecture; this is an important limitation of the study.

Future research may address the following questions:

1. Can a general formula for the sum S be derived for Twin-Prime Graphs that do not satisfy the Eulerian condition?

2. What are the conditions for a Twin-Prime Graph to be Hamiltonian, and how do these conditions relate to the distribution of twin primes?

3. Can a similar additive regularity be demonstrated for prime-triplet or Sophie Germain prime graphs?

Ethics Statement and Additional Information

Ethics Committee Approval: This study is a theoretical/analytical mathematics investigation that does not involve any experimentation on human or animal subjects; therefore, ethics committee approval is not required.

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