# Zeno Arc Constant

## Λ (Capital Lambda)

*A New Irrational Geometric Constant Generated by an Infinite Sequence of Chords*

**İsmail Kara**
*Kocaeli, Türkiye*
ismailkarailetisim@gmail.com | ismailkara.com.tr
ORCID: 0009-0009-5651-0193

| | |
|---|---|
| **Symbol** | Λ (capital Lambda) |
| **Rationale for Symbol** | Evokes angles and chords in geometry; the angular form of the letter Lambda visually represents the chord structure. |
| **Approximate Value** | $\Lambda \approx 2.053307\ldots$ radians ($\approx 117.646°$) |
| **Mathematical Type** | Irrational (conjectured; individual terms proven transcendental — see Section 4) |
| **Main Tools** | Niven's Theorem, Lindemann–Weierstrass Theorem |
| **Date** | 2026 |

## Abstract

This study examines the total rotation angle generated by an infinite number of straight line segments placed inside a circle of diameter 1, with chord lengths forming a geometric sequence (1/2, 1/4, 1/8, …). This total angle is named the Zeno Arc Constant (Λ), and its value has been recomputed here at high precision to be approximately 2.053307… radians (≈ 117.646°). Niven's Theorem is used to rigorously show that every individual term of the defining series beyond the first is transcendental; whether this makes the full infinite sum Λ itself irrational, however, is not established by the argument in the original draft (see Section 4.3), and this English revision presents that conclusion as an open conjecture rather than a proven theorem. Possible applications in discrete-spacetime models in quantum mechanics and in robotic path-planning algorithms are also discussed.

**Keywords:** *Zeno Arc Constant, irrational number, arcsin series, Niven's Theorem, transcendental number, geometric chord sequence*

## 1. Introduction and Motivation

Mathematical constants are objects that often open the door to deep analytic structures, including those arising from apparently simple geometric questions. The emergence of π from the area of a circle and of *e* from compound interest are classic examples of this phenomenon. The Zeno Arc Constant (Λ) introduced in this study has a similar geometric starting point: inspired by Zeno's paradox of infinite division, it examines the total rotation angle produced by an infinite number of straight line segments arranged so that their lengths form a geometric sequence.

The original contribution of this study is concentrated in three points: (i) providing a closed-form definition of the constant Λ; (ii) examining its irrationality using Niven's Theorem and the Lindemann–Weierstrass principle; and (iii) discussing possible physical and algorithmic applications of the constant.

## 2. Geometric Construction

### 2.1 The Circle and the Chord Sequence

Consider a circle of diameter 1 unit, so that the radius is $r = 1/2$ unit. Let the topmost point of the circle (the "north pole," K) be our starting point. We construct the following infinite sequence of chords:

- Step 1: Starting from point K, a chord of length $\ell_1 = 1/2$ is drawn; this chord meets the circle at point $Q_1$.

- Step 2: From point $Q_1$, a second chord of length $\ell_2 = 1/4$ is drawn; it meets the circle at point $Q_2$.

- Step n: From point $Q_{n-1}$, the *n*-th chord, of length $\ell_n = 1/2^n$, is drawn; it meets the circle at point $Q_n$.

The total distance traveled is:

$$\Sigma_{n=1}^{\wedge}\infty\ (1/2^n) = 1/2 + 1/4 + 1/8 + \cdots = 1\ (unit)$$

That is, the cumulative chord length is exactly equal to the diameter of the circle. However, since the path traveled is not a straight line, whether the south pole is ever reached is a separate question.

### 2.2 The Chord–Angle Relation

In a circle of diameter 1 (radius $r = 1/2$), the central angle subtended by a chord of length $\ell$ is:

$$\theta(\ell) = 2 \arcsin(\ell / 2r) = 2 \arcsin(\ell) \quad [\ell \leq 1, r = 1/2]$$

Applying this relation to each step of the sequence, the angle obtained for each chord is:

$$\theta_n = 2 \arcsin(1 / 2^n)$$

## 3. Definition of the Zeno Arc Constant

The sum of all the chord angles is defined as:

$$\Lambda = \Sigma_{n=1}^{\infty} \theta_n = 2 \arcsin(1/2) + 2 \arcsin(1/4) + 2 \arcsin(1/8) + \cdots = \Sigma_{n=1}^{\infty} 2 \arcsin(1/2^n)$$

### 3.1 Computation of the Approximate Value

The numerical contributions of the first several terms are given in the table below:

| n | $\ell_n = 1/2^n$ | $\theta_n = 2 \arcsin(1/2^n)$ | Cumulative Λ |
|---|---|---|---|
| 1 | 0.5000 | 60.000° (π/3 rad) | 60.000° |
| 2 | 0.2500 | 28.955° | 88.955° |
| 3 | 0.1250 | 14.362° | 103.317° |
| 4 | 0.0625 | 7.168° | 110.485° |
| 5 | 0.0313 | 3.583° | 114.068° |
| 7 | 0.0078 | 0.895° | 116.751° |
| ∞ | → 0 | → 0 | Λ ≈ 117.646° |

The series converges, because for small x the relation $\arcsin(x) \approx x + x^3/6 + \ldots$ shows that the terms of the arcsin series shrink at least as fast as the underlying geometric sequence. High-precision numerical computation gives:

$$\Lambda \approx 2.0533068778\ldots \text{ radians } (\approx 117.6458°)$$

*Remark (Corrected Numerical Value).* Recomputing the series with arbitrary-precision arithmetic (50 decimal digits) through n = 50 shows the sum is fully stable from around n = 30 onward. This refines the original estimate of 2.037676… radians (≈ 116.7°) — which corresponds closely to the partial sum through only about n = 7 (116.751°, see table row above) rather than the fully converged series — to the correct converged value Λ ≈ 2.053307 radians (≈ 117.646°). This correction does not affect the geometric reasoning of Section 2 or the term-by-term analysis of Section 4.

This value is smaller than π, showing that the zigzag path does not reach the south pole and instead comes to rest at an asymmetric, distinguished point on the circle.

## 4. Assessment of Irrationality

This section examines whether $\Lambda$ can be rational, using two tools: Niven's Theorem and the Lindemann–Weierstrass Theorem. As discussed in Section 4.3, these tools rigorously establish that each individual term of the series (beyond the first) is transcendental, but do not, as originally argued, establish that the infinite sum itself is irrational; that conclusion is presented here as an open conjecture.

### 4.1 Niven's Theorem

> **Theorem 1 (Niven, 1956).**
> *In the interval [0°, 90°], the only angles $\theta$ for which both $\theta$ (in degrees) and $\sin\theta$ are rational are 0°, 30°, and 90°.*

Applying this theorem to the terms of the series offers a critical observation. Writing $\varphi_n = \theta_n/2 = \arcsin(1/2^n)$, we have $\sin(\varphi_n) = 1/2^n$, which is rational for every n. For $n = 1$, $\varphi_1 = \arcsin(1/2) = 30°$, one of Niven's special angles, so $\theta_1 = 2\arcsin(1/2) = 60° = \pi/3$ is indeed a rational multiple of $\pi$.

For $n \geq 2$, however, $\sin(\varphi_n) = 1/2^n \in \{1/4, 1/8, 1/16, \ldots\}$, none of which equals 0, 1/2, or 1. By Niven's Theorem, this means $\varphi_n$ cannot be a rational number of degrees for $n \geq 2$ — if it were, together with its rational sine value it would force $\varphi_n \in \{0°, 30°, 90°\}$, which it is not. Hence $\varphi_n/\pi$, and therefore $\theta_n/\pi = 2\varphi_n/\pi$, is irrational for every $n \geq 2$. This is a rigorously established fact for each individual term.

### 4.2 The Lindemann–Weierstrass Theorem

> **Theorem 2 (Lindemann–Weierstrass).**
> *If $\alpha_1, \ldots, \alpha_n$ are algebraic numbers that are linearly independent over the rational numbers $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

*Remark (Corrected Statement).* The statement of this theorem in the original draft was imprecise (referring to independence "over the algebraic numbers" rather than over $\mathbb{Q}$, and phrased as a double negative). The statement above is the standard form of the theorem. A frequently used corollary, established via this theorem, is that for any nonzero algebraic number $\alpha$, the value $\sin(\alpha)$ is transcendental — and consequently, for a nonzero rational value r other than 0, $\pm 1/2$, $\pm 1$, the value $\arcsin(r)$ is transcendental. This corollary, combined with the reasoning of Section 4.1, supports the individual transcendence of $\varphi_n = \arcsin(1/2^n)$ (and hence of $\theta_n$) for every $n \geq 2$.

### 4.3 A Gap Between Individual Transcendence and the Irrationality of the Sum

The original draft's argument proceeded from "each $\theta_n$ ($n \geq 2$) is a transcendental angle" directly to "therefore the infinite sum $\Lambda = \Sigma\theta_n$ is irrational, and $\Lambda/\pi$ is irrational," citing the Lindemann–Weierstrass Theorem as justification.

This step does not follow from the theorem as stated. The Lindemann–Weierstrass Theorem is a precise statement about finitely many exponentials of algebraic numbers; it does not assert, and cannot be used to directly conclude, that an infinite sum of individually transcendental (or individually irrational) quantities must itself be irrational. This is not merely a technicality: the general claim is false as stated. For example, $e$ and $(5 - e)$ are both individually transcendental (if $5 - e$ were algebraic, $e = 5 - (5 - e)$ would be algebraic too, a contradiction), yet their sum $e + (5 - e) = 5$ is rational. Sums of transcendental or irrational numbers can be rational, algebraic, or transcendental depending on delicate relationships between the terms — relationships that are not addressed by a general appeal to Lindemann–Weierstrass.

Consequently, while the transcendence of each individual term $\theta_n$ ($n \geq 2$) is rigorously established, the irrationality of the full series $\Lambda = \Sigma\theta_n$ is not established by the argument given, and is presented here as an open conjecture rather than a proven theorem.

### 4.4 Conjecture and Supporting Considerations

*Conjecture 4.1 (Irrationality of Λ).* $\Lambda$ is conjectured to be irrational and not a rational multiple of $\pi$.

This conjecture is plausible for the following reasons, though neither constitutes a proof: (i) the terms $\theta_n$ ($n \geq 2$) are individually transcendental and arise from algebraically unrelated arguments (1/4, 1/8, 1/16, …), making an exact cancellation to a rational sum appear structurally unlikely, though not impossible without a specific mechanism ruling it out; (ii) the numerically computed value $\Lambda \approx 2.0533068778\ldots$ shows no evident closed-form relationship to $\pi$ ($\Lambda/\pi \approx 0.653588\ldots$ does not match any simple low-denominator fraction). Establishing the conjecture rigorously — for instance, via Baker's theorem on linear forms in logarithms, or a bespoke argument exploiting the specific geometric structure of the series — remains an open problem (see Section 6).

## 5. Possible Application Areas

*As with the original study, the applications below are speculative and intended as conceptual bridges rather than established physical results.*

### 5.1 Discrete Spacetime Curvature in Quantum Mechanics

While General Relativity models spacetime as smooth and continuous, various approaches to quantum gravity propose that spacetime may have a discrete (pixel-like) structure at the Planck scale (see Loop Quantum Gravity, LQG). In this context, one might imagine that a photon travelling along a curved trajectory near a black hole's cross-section does not trace a smooth circle but instead forms linear segments on the order of the Planck length. If the lengths of these segments form a geometric sequence, the deflection angle could be expressed in terms of the constant Λ; this offers a speculative point of possible connection.

### 5.2 Robotics and Algorithm Optimization

For an autonomous robot moving within a circular workspace that halves its energy at every step (Zeno-type energy consumption), the constant Λ could serve as a direct reference value for computing the robot's eventual fixed stopping point. The blind-spot coordinate can be parametrized by Λ as:

$$(x, y) = ( r \cos(\Lambda), r \sin(\Lambda) ) \quad [r = radius]$$

The practical significance of this application lies in determining the optimality bounds of path-planning algorithms operating under discrete energy constraints.

## 6. Open Questions

This study leaves the following questions open:

1. Is Λ a Liouville number, or does it possess stronger transcendence properties?

2. What are the Mahler measure or irrationality-exponent values of the terms $\arcsin(1/2^n)$?

3. What analytic properties characterize the family of constants obtained by varying the geometric weights of the sequence ($q^n$ in place of $1/2^n$, for $0 < q < 1$)?

4. Does the continued fraction expansion of Λ exhibit any discernible pattern?

5. **(New)** Can the irrationality of Λ (Conjecture 4.1) be established rigorously, closing the gap identified in Section 4.3 between the transcendence of the individual terms $\theta_n$ and the status of their infinite sum?

## 7. Conclusion

In this study, the Zeno Arc Constant Λ ≈ 2.053307… (corrected from the original estimate of 2.037676…) has been defined via the total rotation angle produced by a sequence of geometric chords. Using Niven's Theorem, it has been rigorously shown that every term of the defining series beyond the first is transcendental. Whether this makes the full infinite sum Λ irrational, however, was not established by the original Lindemann–Weierstrass-based argument, which contained an unjustified logical step; this English revision corrects that step and presents the irrationality of Λ as a well-motivated open conjecture rather than a proven result. Potential application areas for the constant in quantum spacetime models and robotic algorithms have been proposed.

Λ appears to occupy an original position in the mathematical literature as the natural accumulation of an infinite geometric process that cannot obviously be expressed as an integer or a rational multiple of π — though, as discussed, a rigorous proof of even its irrationality remains open. A more thorough investigation of the analytic properties of this constant is left to future work.

## References


Borwein, J., & Crandall, R. (2013). Closed forms: What they are and why we care. *Notices of the AMS*, 60(1), 50–65.

Conway, J. H., & Guy, R. K. (1996). *The Book of Numbers*. Springer-Verlag.

Lindemann, F. (1882). Über die Zahl π. *Mathematische Annalen*, 20, 213–225.

Niven, I. (1956). *Irrational Numbers* (Carus Monograph No. 11). The Mathematical Association of America.

Weierstrass, K. (1885). Zu Lindemann's Abhandlung: Über die Ludolph'sche Zahl. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften*, 2, 1067–1086.